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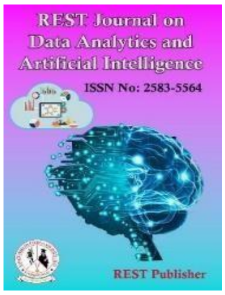

Strong Implicative Filters of Residuatedlattice Wajsberg Algebras

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Abstract: *In aim of this paper is to give some characterization of fuzzy positive and associative implicative filters of residuatedlattice Wajsberg algebra, and some properties are studied with respect to fuzzy subset. We establish a set of equivalent condition for every fuzzy positive and associative implicative filters become an implicative filter of residuated lattice Wajsberg algebra.*

***Keywords**: Wajsberg algebra; Lattice Wajsberg algebra; Residuated lattice Wajsberg algebra;Strong implicative filter.*

1. INTRODUCTION

Residuated lattices were introduced by Ward and Dilworth [10]. Mordchaj Wajsberg introduced the concept of Wajsberg algebras in [9].Y.Xu and K.Qin [11] introduced the concepts of filter and implicative filter in a lattice implication algebra. M.Basheer Ahamed and A.Ibrahim [1,2, 4] introduced the definitions of fuzzy implicative filter, anti-fuzzy implicative filter, positive and associative implicative filteroflattice Wajsberg algebras and obtained some properties. Recently, the authors [5, 6, 7, 8] introduced the notions of implicative filter, fuzzy implicative filter, ananti fuzzy implicative filter and positive implicative filter of residuated lattice Wajsberg algebras and discussed some of their properties.

In this research paper, we introduce the notion of strong implicative filter of residuated lattice Wajsberg algebra. Also, we obtain some of their related properties. Finally, we discuss some equivalent conditions with illustrations.

2. PRELIMINARIES

We recollect some basic definitions and results which will be used in the main results.

Definition 2.1[3] Let $(A, \to, *, 1)$ be a algebra with quasi complement "*" and a binary operation "→". Then it is called a Wajsberg algebra, if the following axioms are satisfied for all $x, y, z \in A$,

(i) $1 \to x = x$
(ii) $(x \to y) \to ((y \to z) \to (x \to z)) = 1$
(iii) $(x \to y) \to y = (y \to x) \to x$
(iv) $(x^* \to y^*) \to (y \to x) = 1$.

Proposition 2.2[3]. Let$(A, \to, *, 1)$ be a Wajsberg algebra. Then the following axioms are satisfied for all$x, y, z \in A$,

(i) $x \to x = 1$
(ii) If $(x \to y) = (y \to x) = 1$ then $x = y$.
(iii) $x \to 1 = 1$
(iv) $(x \to (y \to x)) = 1$
(v) If $(x \to y) = (y \to z) = 1$ then $x \to z = 1$
(vi) $(x \to y) \to ((z \to x) \to (z \to y)) = 1$

(vii) $x \to (y \to z) = y \to (x \to z)$
(viii) $x \to 0 = x \to 1^* = x^*$
(ix) $(x^*)^* = x$
(x) $(x^* \to y^*) = y \to x$.

Definition 2.3 [3]. Let $(A, \to, *, 1)$ be Wajsberg algebra. Thenitis called a lattice Wajsberg algebra, if the following conditions are satisfied for all $x, y \in A$,

(i) The partial ordering $\leq$ on a lattice Wajsberg algebra A such that $x \leq y$ if and only if $x \to y = 1$
(ii) $(x \vee y) = (x \to y) \to y$
(iii) $(x \wedge y) = ((x^* \to y^*) \to y^*)^*$.

Thus $(A, \vee, \wedge, *, \to, 0, 1)$ is a lattice Wajsberg algebra with lower bound 0 and upper bound 1.

Proposition 2.4[3]. Let$(A, \to, *, 1)$ be a lattice Wajsberg algebra. Then the following axioms are satisfiedfor all $x, y, z \in A$,

(i) If $x \leq y$ then $x \to z \geq y \to z$ and $z \to x \leq z \to y$
(ii) $x \leq y \to z$ if and only if $y \leq x \to z$
(iii) $(x \vee y)^* = (x^* \wedge y^*)$
(iv) $(x \wedge y)^* = (x^* \vee y^*)$
(v) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$
(vi) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$
(vii) $(x \to y) \vee (y \to x) = 1$
(viii) $x \to (y \vee z) = (x \to y) \vee (x \to z)$
(ix) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$
(x) $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$
(xi) $(x \wedge y) \to z = (x \to y) \to (x \to z)$.

Definition 2.5 [10]. Let$(A, \vee, \wedge, *, \to, 0, 1)$ be an algebra of type (2,2,2,2,0,0). Then it is called a residuated lattice, if the following axioms are satisfied for all $x, y, z \in A$,

(i) $(A, \vee, \wedge, 0, 1)$is a bounded lattice
(ii) $(A, \odot, 1)$is a commutative monoid
(iii) $x \odot y \leq z$if and only if $x \leq y \to z$

Definition 2.6[5] Let $(A, \vee, \wedge, *, \to)$be a lattice Wajsberg algebra. Then it is called a residuated lattice Wajsberg algebras, if a binary operation "$\odot$"on $A$is satisfied the condition $x \odot y = (x \to y^*)^*$for all$x, y \in A$.

Note. From the definition 2.6, we have$(A, \vee, \wedge, \odot, \to, *, 0, 1)$ is called a residuated lattice Wajsberg algebra.

Definition 2.7 [1] Let A be a latticeWajsberg algebra. Then a subset F of A is called an implicative filter of A,if the following axioms are satisfied for all $x, y \in A$,

(i) $1 \in F$
(ii) $x \in F$and$x \to y \in F$imply $y \in F$.

Proposition 2.8[1] Let A be a residuated lattice Wajsberg algebra and let $M$and N be a filters of A , $M \subseteq N$. If M is an implicative filter then N is also an implicative filter.

Proposition 2.9 [5]Let Fbe an implicative filter in residuated lattice Wajsberg algebra and let $x \in F$ and $x \leq y$, then $y \in F$.

3. PROPERTIES OF STRONGIMPLICATIVE FILTER

In this section, we introducestrong implicative filter of residuated lattice Wajsbergalgebra and obtain some useful results with illustrations.

Definition 3.1. Let A be a residuated latticeWajsberg algebra. A subset F of A is called a strong implicative filter of A if it satisfies the following axioms for all $x, y, z \in A$,

(i) $1 \in F$

(ii)If $x, y \in F$ then $x \odot y \in F$

(iii) $x \to (y \to z) \in F$ and $x \to y \in F$ imply $x \to z \in F$.

Example 3.2. Let $A = \{0, i, j, k, l, m, n, o, p, 1\}$be a set with Figure (3.1) as a partial ordering. Define abinary operation "→"and aquasi-complement "*"on$A$ as in tables (3.1) and (3.2).

Define ∨ ,∧ and ⊙operations on $(A, \vee, \wedge, 0, 1)$ as follows

$(x \vee y) = (x \to y) \to y$

$$(x \wedge y) = ((x^* \to y^*) \to y^*)^*$$

$x \odot y = (x \to y^*)^*$forall $x, y \in A$.

Then $(A, \vee, \wedge, \odot, \to, *\, 0, 1)$ is a residuated lattice Wajsberg algebra.

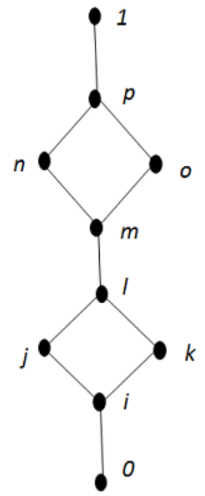

FIGURE 1. Lattice diagram

TABLE 1. Complement

x	x^*
0	1
i	l
j	k
k	j
l	i
m	p
n	o
o	n
p	m
1	0

TABLE 2. Implication

$\rightarrow$	0	i	j	k	l	m	n	o	p	1
0	1	1	1	1	1	1	1	1	1	1
i	p	1	1	1	1	1	1	1	1	1
j	o	o	1	o	1	1	1	1	1	1
k	n	n	n	1	1	1	1	1	1	1
l	m	m	n	o	1	1	1	1	1	1
m	l	l	l	l	l	1	1	1	1	1
n	j	j	j	l	l	o	1	o	1	1
o	k	k	l	k	l	n	n	1	1	1
p	i	i	j	k	l	m	n	o	1	1
1	0	i	j	k	l	m	n	o	p	1

Consider a fuzzy subset μ on A is defined by $\mu(x) = \begin{cases} 0.7 & if\ x = 1 \\ 0.4 & other\ wise \end{cases}$ for all $x \in A$Then, we have μ is a fuzzy positive implicative filter of residuated lattice Wajsberg algebra A.

Consider a fuzzy subset μon A is defined by$\mu(x) = \begin{cases} 0.8 & if\ x \in 0, j, k, l \\ 0.2 & if\ x \in i, m, n, o, p, 1 \end{cases}$ for all $x \in A$.Then, we have μ is not a fuzzy positive implicative filter of residuated lattice Wajsberg algebra A. Since $\mu(0 \to j) \ngeq \min\{\mu((i \odot 0) \to j), \mu(0 \to i)\}$.

Proposition 3.3 Every strong implicative filter of a residuated lattice Wajsberg algebra is animplicative filter.

Proof.Let F be strong implicative filter of residuated lattice Wajsberg algebra and $x \to y \in F$ and $x \in F$ for all $x, y \in A$.

Replace z by y in definition 3.1.

Then, we have $x \to (y \to z) = x \to (y \to y)$

$$= x \to 1$$

$$= x \in F.$$

Also, $x \to y \in F$. Then, we get$x \to z \in F$.

Hence, every strong implicative filter of a residuated lattice Wajsberg algebra is an implicative filter.

Proposition 3.4 Let$F$ be an strong implicative filter of a residuated lattice Wajsberg algebra such that $x \to (y \to (y \to z)) \in F$ and $x \in F$ imply $x \to z$ for all $x, y, z \in A$. Then F is an strong implicative filter of A.

Proof.Let $x \to (y \to (y \to z)) \in F$ and $x \to y \in F$ for all $x, y, z \in A$.

From (vii) of definition 2.2 and (ii) of definition 2.1, we have

$x \to (y \to z) = y \to (x \to z) \leq (x \to y) \to (x \to (x \to z))$

$(x \to y) \to (x \to (x \to z)) \in F$ [From proposition 2.9]

Since, $x \to y$, we have $x \to z \in F$.

Hence, F is a strong implicative filterof A.

Definition 3.5. Let A be a residuated latticeWajsberg algebra, $a \in A$.The interval $[a, 1]$ defined as $[a, 1] = \{x/x \in A, a \leq x\}$ of A denoted as $I(a)$.

Proposition 3.6. Let A be a residuated latticeWajsberg algebra, $a \in A$, then $\{1\}$ is strong implicative filter of A if and only if $I(a)$ is an implicative filter of A for any $a \in A$.

Proof. Suppose that $\{1\}$ is strong implicative filter of A. For any $a \in A, 1 \in A$ is trivial. If $x \in I(a)$ and $x \to y \in I(a)$, then $a \leq x, a \leq x \to y$, that is $a \to x = 1 \in \{1\}$ and $a \to (x \to y) = 1 \in \{1\}$.It follows that $a \to y \in \{1\}, a \leq y$, and hence $y \in I(a)$. Thus, $I(a)$ is an implicative filter of A.

Conversely, assume that $I(a)$ is an implicative filter of A for any $a \in A$.For any $xy, z \in A$, if $x \to (y \to z) \in \{1\}$ and $x \to y \in \{1\}$, then $x \leq y \to z, x \leq y$, it follows that $x \leq z$ because $I(x)$ is an implicative filter, hence $x \to z = 1 \in \{1\}$.Hence, $\{1\}$ is strong implicative filter of A.

Proposition 3.7. Let$A$ be a residuated latticeWajsberg algebra, $F \subseteq A$. The following statements are equivalent

(i) F is strong implicative filter

(ii) F is an implicative filter and for any $x, y \in A, x \to (x \to y) \in F$ implies $x \to y \in F$

(iii) F is an implicative filter and for any $x, y, z \in A, x \to (y \to z) \in F$implies$(x \to y) \to (x \to z) \in F$

(iv) $1 \in F$and for any $x, y, z \in A, z \to (x \to (x \to y)) \in F$ and $z \in F$ imply $x \to y \in F$.

Proof.

(i) ⇒ (ii)

For any $x, y \in A$, if $x \to (x \to y) \in F$, since $x \to x = 1 \in F$, from (ii) of definition 3.1 we have $x \to y \in F$.

(ii) ⇒ (iii)

Assume that (ii) holds. For any $x, y, z \in A$, suppose $x \to (y \to z) \in F$, from (vi) ,(vii) of proposition 2.2 and (i), (ii) of proposition 2.3 $x \to (y \to z) \leq x \to \big((x \to y) \to (x \to z)\big)$.Therefore from proposition 2.9 and (vii) of proposition 2.2, we get $x \to \Big(x \to \big((x \to y) \to z\big)\Big) = x \to \big((x \to y) \to (x \to z)\big) \in F$.

$x \to \big((x \to y) \to z\big) = (x \to y) \to (x \to z) \in F$ [From (ii) and (vii) of proposition 2.2].

(iii) ⇒(iv)

Assume that (iii) holds and we prove (iv). Since, $1 \in F$ is trivial. For any $x, y, z \in A$, suppose $z \to (x \to (x \to y)) \in F$ and $z \in F$, then $x \to (y \to z) \in F$. From (ii) of definition 2.7, we have $x \to y = 1 \to (x \to y) = (x \to x) \to (x \to y) \in F$.

(iv) ⇒(i)

Suppose $x \in F$ and $x \to y \in F$, then we get $x \to \big(1 \to (1 \to y)\big) = x \to y \in F$, it follows that $y = 1 \to y \in F$ and hence F is an implicative filter. For any $x, y, z \in A, x \to (y \to z) \in F$ and $x \to y \in F$,

Now, $\big(x \to (y \to z)\big) \to \Big((x \to y) \to \big(x \to (x \to z)\big)\Big)$

$$= (x \to y) \to (\big(x \to (y \to z)\big) \to \big(x \to (x \to z)\big))$$

$$= (x \to y) \to (\big(y \to (x \to z)\big) \to \big(x \to (x \to z)\big))$$

$$= (x \to y) \to (x \to \big(y \vee (x \to z)\big))$$

$$= (x \to y) \to \Big((x \to y) \vee \big(x \to (x \to z)\big)\Big) = 1 \in F.$$

It follows that $(x \to y) \to \big(x \to (x \to z)\big) \in F$ and hence $x \to z \in F$ by $x \to y \in F$ and (iv).

Proposition 3.8. Let$A$ be a residuated latticeWajsberg algebra, F_1 and F_2 are any two implicative filters of $A, F_1 \subseteq F_2$. If F_1 is a strong implicative filter, so is F_2.

Proof. Suppose $x \rightarrow (x \rightarrow y) \in F_2$, we only to prove $x \rightarrow y \in F_2$.

Now $x \rightarrow \Big(x \rightarrow \big(\big(x \rightarrow (x \rightarrow y)\big) \rightarrow y\big)\Big) = \big(x \rightarrow (x \rightarrow y)\big) \rightarrow \big(x \rightarrow (x \rightarrow y)\big) = 1 \in F_1$.

It follows that $x \rightarrow ((x \rightarrow (x \rightarrow y)) \rightarrow y \in F_1 \subseteq F_2$.

That is, $(x \rightarrow (x \rightarrow y)) \rightarrow (x \rightarrow y) \in F_2$ and hence $x \rightarrow y \in F_2$. Hence F_2 is a strong implicative filter of A.

5. CONCLUSION

We have proposed the notionof strong implicative filter of residuated lattice Wajsberg algebra.Furthermore, we discussed an equivalent condition that everystrong implicative filter is an implicative filter. We also derived some properties of strong implicative filter of residuated lattice Wajsberg algebra.

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