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Enhanced Mathematical Models for the Sombor Index: Reduced and Co-Sombor Index Perspectives

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Abstract
In this study, we extend the mathematical framework of the Sombor index and its variants by developing enhanced versions: The Enhanced Sombor Index (ESO), Enhanced Reduced Sombor Index (ERSO), and Enhanced co-Sombor Index (ECSO). These enhanced indices incorporate weighting functions to provide a more nuanced analysis of graph properties. We derive key properties and theorems, demonstrating that the enhanced indices are at least as large as their traditional counterparts. We also establish upper bounds for these indices in bipartite graphs, specifically $K_{3,3}$. Practical applications in chemical graph theory, social network analysis, and biological networks illustrate the utility of these enhanced indices. Detailed calculations for the complete bipartite graph $K_{3,3}$ validate our theoretical findings and demonstrate the practical computation of the indices. Potential future research directions include generalization to other graph classes, optimization of weighting functions, algorithmic development, application to dynamic networks, empirical validation, and interdisciplinary applications.
Keywords: Graph Theory, Sombor Index, Enhanced Sombor Index, Enhanced Reduced Sombor Index, Reduced Sombor Index, co-Sombor Index, Enhanced co-Sombor Index.

1. Introduction

Background on Graph Theory and the Significance of Graph Invariants:

Graph theory is an important field of discrete mathematics which studies the graphs, that are mathematical structures used to model pairwise relations between objects. A graph $G = (V, E)$ consists of a set V, which contains vertices and a set E containing edges that connect two individual nodes. Graph invariants are very important because they give ways to distinguish between different types of graphs by producing the same answers for each pair which isomorphic. Some of the examples are degree sequence, chromatic number and topological indices (West 2001).

Topological indexes are numerical values of a graph corresponding to its topology. These indices are used in many other areas: chemistry (exploring molecular structures), computer science (network analysis) or sometimes even completely different fields trying to model a complex system. Of the two, distance-based and degree-based indices have become popular because they can be easily computed (Gutman & Trinajstić 1972).

Introduction to the Sombor Index (SO), Reduced Sombor Index (RSO), and co-Sombor Index (CSO):

Gutman (2021) presented a novel degree-based topological index - the Sombor Index(SO) for short, given to G as;

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2}$$

d_u, d_v (degrees of vertices u, v). It captures the correlation between degrees of pairs vertices and was proved to be correlated to differ thermodinamic physico-chemical properties as well.

By using the Sombor Index as a base, we define and express in normalized degree contributions for every node with Reduced Sombor Index (RSO) form:

$$RSO(G) = \sum_{uv \subset E(G)} \frac{\sqrt{d_u^2 + d_v^2}}{d_u + d_v}$$

Normalization to ensure the index is independent of protein size gives a value that can incorporate information about what quantities adjacent vertices contribute.

Another variant on this - the co-Sombor Index (CSO) adjusts these degrees by subtracting one, producing a parameter that focuses more upon connectedness of a graph:

$$\text{CSO}\ (G) = \sum_{uv \in E(G)} \sqrt{(d_u - 1)^2 + (d_v - 1)^2}$$

This change shifts once attention to local information of its own design, and that in turn enables understanding the connectivity properties of a graph.

2. Objective of the Paper:

The main purpose of this chapter is to design new improved mathematical models regarding Sombor Index and its related types as the Reduced Sombor Index and co-Sombor index. We hope that adding more weighting functions and theoretical refinements will:

a) Enhance the discriminatory power and applicability of these indices in complex network analysis.

b) Derive new mathematical properties and theorems that provide deeper insights into the behavior of these indices.

c) Present computational techniques and practical applications to demonstrate the effectiveness of the enhanced indices.

This study contributes to the ongoing research in graph theory by offering advanced tools for the analysis of graph invariants and their applications in various scientific and engineering domains.

3. Preliminaries

Definition of Basic Graph Theoretical Terms and Notations:

Let $G = (V, E)$ be a simple, connected graph where $V(G)$ represents the set of vertices and $E(G)$ represents the set of edges. The number of vertices $|V(G)|$ is denoted by n, and the number of edges $|E(G)|$ is denoted by m.

- Vertex Degree: The degree of a vertex $v \in V(G)$, denoted by d_v, is the number of edges incident to v. Mathematically,

$$d_v = \sum_{u \in V(G)} a_{vu}$$

where a_{vu} is the adjacency matrix element, which is 1 if u and v are adjacent, and 0 otherwise (West, 2001).

- Path and Distance: A path P in G is a sequence of vertices $(v_1, v_2, \ldots, v_k)$ such that v_i is adjacent to v_{i+1} for $1 \leq i < k$. The distance $d(u, v)$ between two vertices u and v is the length of the shortest path connecting them (Harary, 1969).

Graph Invariants: Properties of a graph that remain unchanged under isomorphisms. Examples include the diameter, radius, and various topological indices (Bondy & Murty, 1976).

Introduction to the Mathematical Formulation of the Sombor Index:

The Sombor Index (SO) is a degree-based topological index defined to capture the structural properties of a graph based on vertex degrees. It is given by:

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2}$$

where d_u and d_v are the degrees of vertices u and v in G (Gutman, 2021).

Detailed Mathematical Formulation:

1. Sombor Index (SO):

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2}$$

This index sums the square root of the sum of squares of degrees of adjacent vertices. It highlights the combined influence of vertex degrees in edge contributions.

2. Reduced Sombor Index (RSO):

$$RSO(G) = \sum_{uv \in E(G)} \frac{\sqrt{d_u^2 + d_v^2}}{d_u + d_v}$$

Here, the index normalizes the degree contributions by dividing by the sum of degrees. This provides a relative measure that balances the degree contributions (Gutman, 2021).

3. co-Sombor Index (CSO):

$$CSO(G) = \sum_{uv \in E(G)} \sqrt{(d_u - 1)^2 + (d_v - 1)^2}$$

This variant modifies the degrees by subtracting one, emphasizing the connectivity and interaction patterns of the graph (Gutman, 2021).

Mathematical Properties:

- ***Monotonicity***: The Sombor Index $SO(G)$ increases with the addition of edges or vertices with higher degrees, indicating its sensitivity to graph density.
- ***Bounds***: For a graph G with minimum degree δ and maximum degree Δ, the following bounds hold:

$$\sqrt{2}m\delta \leq SO(G) \leq \sqrt{2}m\Delta$$

where m is the number of edges in G (Gutman, 2021).

- ***Relationship with Other Indices***: The Sombor Index is related to other degree-based indices such as the Zagreb indices M_1 and M_2 :

$$M_1(G) = \sum_{v \in V(G)} d_v^2,\ M_2(G) = \sum_{uv \in E(G)} d_u d_v$$

The Sombor Index can be viewed as a geometric mean counterpart to these indices (Gutman & Trinajstić, 1972).

These schemes are defined on the background level, and knowing their mathe- From these index definitions to study of such properties we can be visualized as a preparation step for improving some reasonable notion of structural descriptor that captures the full acyclic structure in general. It is the purpose of this paper to suggest improvements and examine their theoretical as well as practical implications.

4. Properties and Theorems

Theorem 1: Enhanced Sombor Index (ESO) Bound

Theorem 1: For a connected graph G with minimum degree δ and maximum degree Δ, the Enhanced Sombor Index ESO (G) satisfies:

$$\text{ESO}\ (G) \geq \sum_{uv\in E(G)} \sqrt{d_u^2 + d_v^2}$$

$$where, \quad \text{ESO}(G) = \sum_{uv\in E(G)} \sqrt{d_u^2 + d_v^2} \cdot f(d_u, d_v) \quad \text{and} \quad f(d_u, d_v) \geq 1$$

Proof: Given,

$$\text{ESO}\ (G) = \sum_{uv\in E(G)} \sqrt{d_u^2 + d_v^2} \cdot f(d_u, d_v)$$

where $f(d_u, d_v)$ is a non-negative function such that $f(d_u, d_v) \geq 1$.

To prove:

$$\text{ESO}\ (G) \geq \sum_{uv\in E(G)} \sqrt{d_u^2 + d_v^2}$$

Step-by-Step Proof:

Start by writing the expression for ESO (G) :

$$\text{ESO}\ (G) = \sum_{uv\in E(G)} \sqrt{d_u^2 + d_v^2} \cdot f(d_u, d_v)$$

Since $f(d_u, d_v) \geq 1$ for all $u, v \in V(G)$, we can assert:

$$\sqrt{d_u^2 + d_v^2} \cdot f(d_u, d_v) \geq \sqrt{d_u^2 + d_v^2}$$

Sum the inequalities over all edges $uv \in E(G)$:

$$\sum_{uv\in E(G)} \sqrt{d_u^2 + d_v^2} \cdot f(d_u, d_v) \geq \sum_{uv\in E(G)} \sqrt{d_u^2 + d_v^2}$$

By definition, the left-hand side is ESO (G) and the right-hand side is $SO(G)$:

$$\text{ESO}\ (G) \geq SO(G)$$

$$\text{Thus, we have proven that } \text{ESO}(G) \geq \sum_{uv\in E(G)} \sqrt{d_u^2 + d_v^2}, \text{ as required}$$

Theorem 2: Enhanced Reduced Sombor Index (ERSO) Bound

Theorem 2: For a connected graph G, the Enhanced Reduced Sombor Index ERSO (G) satisfies:

$$\text{ERSO}\ (G) \geq \sum_{uv\in E(G)} \frac{\sqrt{d_u^2 + d_v^2}}{d_u + d_v}$$

$$\text{where,} \quad \text{ERSO}(G) = \sum_{uv\in E(G)} \frac{\sqrt{d_u^2 + d_v^2}}{d_u + d_v} \cdot g(d_u, d_v) \quad \text{and} \quad g(d_u, d_v) \geq 1$$

Proof: Given,

$$\text{ERSO}\ (G) = \sum_{uv\in E(G)} \frac{\sqrt{d_u^2 + d_v^2}}{d_u + d_v} \cdot g(d_u, d_v)$$

where $g(d_u, d_v)$ is a non-negative function such that $g(d_u, d_v) \geq 1$.

To prove:

$$\text{ERSO}\ (G) \geq \sum_{uv \in E(G)} \frac{\sqrt{d_u^2 + d_v^2}}{d_u + d_v}$$

Step-by-Step Proof:

Start by writing the expression for ERSO (G) :

$$\text{ERSO}\ (G) = \sum_{uv \in E(G)} \frac{\sqrt{d_u^2 + d_v^2}}{d_u + d_v} \cdot g(d_u, d_v)$$

Since $g(d_u, d_v) \geq 1$ for all $u, v \in V(G)$, we can assert:

$$\frac{\sqrt{d_u^2 + d_v^2}}{d_u + d_v} \cdot g(d_u, d_v) \geq \frac{\sqrt{d_u^2 + d_v^2}}{d_u + d_v}$$

Sum the inequalities over all edges $uv \in E(G)$:

$$\sum_{uv \in E(G)} \frac{\sqrt{d_u^2 + d_v^2}}{d_u + d_v} \cdot g(d_u, d_v) \geq \sum_{uv \in E(G)} \frac{\sqrt{d_u^2 + d_v^2}}{d_u + d_v}$$

By definition, the left-hand side is ERSO (G) and the right-hand side is $RSO(G)$:

$$\text{ERSO}\ (G) \geq RSO(G)$$

$$\text{Thus, we have proven that ERSO}(G) \geq \sum_{uv \in E(G)} \frac{\sqrt{d_u^2 + d_v^2}}{d_u + d_v}, \text{as required}$$

Theorem 3: Enhanced co-Sombor Index (ECSO) Bound

Theorem 3: For a connected graph G, the Enhanced co-Sombor Index ECSO (G) satisfies:

$$\text{ECSO}\ (G) \geq \sum_{uv \in E(G)} \sqrt{(d_u - 1)^2 + (d_v - 1)^2}$$

$$where\ \ \text{ECSO}(G) = \sum_{uv \in E(G)} \sqrt{(d_u - 1)^2 + (d_v - 1)^2} \cdot h(d_u, d_v)\ and\ h(d_u, d_v) \geq 1$$

Proof: Given,

$$\text{ECSO}\ (G) = \sum_{uv \in E(G)} \sqrt{(d_u - 1)^2 + (d_v - 1)^2} \cdot h(d_u, d_v)$$

where $h(d_u, d_v)$ is a non-negative function such that $h(d_u, d_v) \geq 1$.

To prove:

$$\text{ECSO}\ (G) \geq \sum_{uv \in E(G)} \sqrt{(d_u - 1)^2 + (d_v - 1)^2}$$

Step-by-Step Proof:

Start by writing the expression for ECSO (G) :

$$\text{ECSO}\ (G) = \sum_{uv \in E(G)} \sqrt{(d_u - 1)^2 + (d_v - 1)^2} \cdot h(d_u, d_v)$$

Since $h(d_u, d_v) \geq 1$ for all $u, v \in V(G)$, we can assert:

$$\sqrt{(d_u - 1)^2 + (d_v - 1)^2} \cdot h(d_u, d_v) \geq \sqrt{(d_u - 1)^2 + (d_v - 1)^2}$$

Sum the inequalities over all edges $uv \in E(G)$:

$$\sum_{uv\in E(G)} \sqrt{(d_u-1)^2+(d_v-1)^2}\cdot h(d_u,d_v) \geq \sum_{uv\in E(G)} \sqrt{(d_u-1)^2+(d_v-1)^2}$$

By definition, the left-hand side is ECSO (G) and the right-hand side is $CSO(G)$:

$$\text{ECSO }(G) \geq \text{CSO }(G)$$

$$\text{Thus, we have proven that } \text{ECSO}(G) \geq \sum_{uv\in E(G)} \sqrt{(d_u-1)^2+(d_v-1)^2}, \text{as required.}$$

Theorem 4: Enhanced Sombor Indices in Bipartite Graphs

Theorem 4: For a connected bipartite graph $G=(U,V,E)$ with vertex sets U and V, the Enhanced Sombor Index ESO (G), Enhanced Reduced Sombor Index ERSO (G), and Enhanced co-Sombor Index ECSO (G) satisfy:

$$\text{ESO }(G) \leq \sqrt{2}m\Delta\cdot \max_{uv\in E(G)} f(d_u,d_v)$$
$$\text{ERSO }(G) \leq \frac{\sqrt{2}m\Delta}{\delta+1}\cdot \max_{uv\in E(G)} g(d_u,d_v)$$
$$\text{ECSO }(G) \leq \sqrt{2}m(\Delta-1)\cdot \max_{uv\in E(G)} h(d_u,d_v)$$

where m is the number of edges, δ is the minimum degree, Δ is the maximum degree, and f, g, and h are non-negative functions defined on vertex degrees.

Proof: We need to establish upper bounds for the enhanced indices, by the given connected bipartite graph $G=(U,V,E)$.

Step-by-Step Proof:

Enhanced Sombor Index (ESO):

The Enhanced Sombor Index ESO (G) is defined as:

$$\text{ESO }(G) = \sum_{uv\in E(G)} \sqrt{d_u^2+d_v^2}\cdot f(d_u,d_v)$$

For a bipartite graph G, the maximum degree Δ is attained by some vertex. Thus, we have:

$$\sqrt{d_u^2+d_v^2} \leq \sqrt{\Delta^2+\Delta^2} = \sqrt{2}\Delta$$

Substituting this into the definition of ESO (G) :

$$ESO(G) = \sum_{uv\in E(G)} \sqrt{d_u^2+d_v^2}\cdot f(d_u,d_v) \leq \sum_{uv\in E(G)} \sqrt{2}\Delta\cdot f(d_u,d_v)$$

Since $f(d_u,d_v)$ is non-negative, we have:

$$\text{ESO }(G) \leq \sqrt{2}\Delta \sum_{uv\in E(G)} f(d_u,d_v)$$

Using the fact that $f(d_u,d_v) \leq \max_{uv\in E(G)} f(d_u,d_v)$:

$$\text{ESO }(G) \leq \sqrt{2}\Delta\cdot m\cdot \max_{uv\in E(G)} f(d_u,d_v)$$

Thus, we have:

$$\text{ESO }(G) \leq \sqrt{2}m\Delta\cdot \max_{uv\in E(G)} f(d_u,d_v)$$

Enhanced Reduced Sombor Index (ERSO):

The Enhanced Reduced Sombor Index ERSO (G) is defined as:

$$\text{ERSO}\,(G)=\sum_{uv\in E(G)}\frac{\sqrt{d_u^2+d_v^2}}{d_u+d_v}\cdot g(d_u,d_v)$$

For $d_u, d_v \geq \delta$, we have:

$$\frac{\sqrt{d_u^2+d_v^2}}{d_u+d_v}\leq\frac{\sqrt{2}\Delta}{\delta+\delta}=\frac{\sqrt{2}\Delta}{2\delta}=\frac{\sqrt{2}\Delta}{\delta+1}$$

Substituting this into the definition of ERSO (G) :

$$\text{ERSO}\,(G)=\sum_{uv\in E(G)}\frac{\sqrt{d_u^2+d_v^2}}{d_u+d_v}\cdot g(d_u,d_v)\leq\sum_{uv\in E(G)}\frac{\sqrt{2}\Delta}{\delta+1}\cdot g(d_u,d_v)$$

Since $g(d_u, d_v)$ is non-negative, we have:

$$\text{ERSO}\,(G)\leq\frac{\sqrt{2}\Delta}{\delta+1}\sum_{uv\in E(G)}g(d_u,d_v)$$

Using the fact that $g(d_u, d_v) \leq \max_{uv\in E(G)} g(d_u, d_v)$:

$$\text{ERSO}\,(G)\leq\frac{\sqrt{2}\Delta}{\delta+1}\cdot m\cdot\max_{uv\in E(G)}g(d_u,d_v)$$

Thus, we have:

$$\text{ERSO}\,(G)\leq\frac{\sqrt{2}m\Delta}{\delta+1}\cdot\max_{uv\in E(G)}g(d_u,d_v)$$

Enhanced co-Sombor Index (ECSO):

The Enhanced co-Sombor Index ECSO (G) is defined as:

$$\text{ECSO}\,(G)=\sum_{uv\in E(G)}\sqrt{(d_u-1)^2+(d_v-1)^2}\cdot h(d_u,d_v)$$

For a bipartite graph G with maximum degree Δ, we have:

$$\sqrt{(d_u-1)^2+(d_v-1)^2}\leq\sqrt{(\Delta-1)^2+(\Delta-1)^2}=\sqrt{2}(\Delta-1)$$

Substituting this into the definition of ECSO (G) :

$$\text{ECSO}(G)=\sum_{uv\in E(G)}\sqrt{(d_u-1)^2+(d_v-1)^2}\cdot h(d_u,d_v)\leq\sum_{uv\in E(G)}\sqrt{2}(\Delta-1)\cdot h(d_u,d_v)$$

Since $h(d_u, d_v)$ is non-negative, we have:

$$\text{ECSO}\,(G)\leq\sqrt{2}(\Delta-1)\sum_{uv\in E(G)}h(d_u,d_v)$$

Using the fact that $h(d_u, d_v) \leq \max_{uv\in E(G)} h(d_u, d_v)$:

$$\text{ECSO}\,(G)\leq\sqrt{2}(\Delta-1)\cdot m\cdot\max_{uv\in E(G)}h(d_u,d_v)$$

Thus, we have:

$$\text{ECSO }(G) \leq \sqrt{2}m(\Delta - 1) \cdot \max_{uv \in E(G)} h(d_u, d_v)$$

These inequalities provide upper bounds for the enhanced Sombor indices in bipartite graphs, showing how these indices are influenced by the maximum degree, the number of edges, and the respective functions f, g, and h.

5. Applications and Examples

Practical Examples and Applications of the Enhanced Indices

The Enhanced Sombor indices, namely the SI index and its Reduced version (RSI), offer a new perspective to calculate more detailed information in different domains such as chemistry, network analysis, biology over conventional measures.

Example 1: Chemical Graph Theory

For instance, think of a chemical molecule as being represented by some graph G in which the vertices indicate atoms and edges show bonds. The Sombor indices have applications in predicting the chemical and biological properties of molecules, such as stability and reactivity.

As one of the topological indices, ESO (G) for chemical molecular graph G with d_i degrees corresponding to atoms, it can be obtained by:

$$\text{ESO }(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2} \cdot f(d_u, d_v)$$

perhaps $f(d_u, d_v)$ is a more abstract function that captures further chemical properties such as electronegativity differences, or bond types.

For example, let:

$$f(d_u, d_v) = \frac{\chi_u + \chi_v}{2}$$

Here, χ_u and χ_v are the given electronegativities of atoms u and v. The ESO for the molecule would be:

$$\text{ESO }(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2} \cdot \frac{\chi_u + \chi_v}{2}$$

Example 2: Network Analysis

A social network is nothing but vertices as individual and edges as relationships. The Enhanced Reduced Sombor Index (ERSI) can also be useful in capturing the connectivity of a network and identifying influential nodes.

For a social network graph G, compute:

$$\text{ERSO }(G) = \sum_{uv \in E(G)} \frac{\sqrt{d_u^2 + d_v^2}}{d_u + d_v} \cdot g(d_u, d_v)$$

where $g(d_u, d_v)$ might account for the strength of the relationship or interaction frequency.

For instance, let:

$$g(d_u, d_v) = \log(d_u d_v + 1)$$

Thus:

$$\text{ERSO }(G) = \sum_{uv \in E(G)} \frac{\sqrt{d_u^2 + d_v^2}}{d_u + d_v} \cdot \log(d_u d_v + 1)$$

Example 3: Biological Networks

As for a protein-protein interaction network, vertices corresponded to proteins while edges represented interactions. The Enhanced co-Sombor Index ECSO reveals important interaction patterns.

For a protein interaction graph G, compute:

$$\mathrm{ECSO}\ (G)=\sum_{uv\in E(G)}\sqrt{(d_u-1)^2+(d_v-1)^2}\cdot h(d_u,d_v)$$

where $h(d_u,d_v)$ might reflect interaction strength or biological significance.

Suppose:

$$h(d_u,d_v)=\frac{k_{uv}}{d_ud_v}$$

where k_{uv} is the interaction strength between proteins u and v. Thus:

$$\mathrm{ECSO}\ (G)=\sum_{uv\in E(G)}\sqrt{(d_u-1)^2+(d_v-1)^2}\cdot\frac{k_{uv}}{d_ud_v}$$

Comparison with Traditional Indices:

As a benchmark tool for showcasing the efficiency of these supplementary indices, we will provide comparison with some traditional tools such as Sombor Index (SO), Zagreb indies - M_1 and M_2 from among which minimum degree corresponds to SO in both cases; Randic index.

Traditional Indices:

1 Sombor Index (SO):

$$SO(G)=\sum_{uv\in E(G)}\sqrt{d_u^2+d_v^2}$$

2 First Zagreb Index (M_1) :

$$M_1(G)=\sum_{v\in V(G)}d_v^2$$

3 Second Zagreb Index (M_2) :

$$M_2(G)=\sum_{uv\in E(G)}d_ud_v$$

4 Randic Index:

$$R(G)=\sum_{uv\in E(G)}\frac{1}{\sqrt{d_ud_v}}$$

Enhanced Indices:

1 Enhanced Sombor Index (ESO):

$$\mathrm{ESO}\ (G)=\sum_{uv\in E(G)}\sqrt{d_u^2+d_v^2}\cdot f(d_u,d_v)$$

2 Enhanced Reduced Sombor Index (ERSO):

$$\mathrm{ERSO}\ (G)=\sum_{uv\in E(G)}\frac{\sqrt{d_u^2+d_v^2}}{d_u+d_v}\cdot g(d_u,d_v)$$

3 Enhanced co-Sombor Index (ECSO):

$$\text{ECSO}\ (G) = \sum_{uv \in E(G)} \sqrt{(d_u - 1)^2 + (d_v - 1)^2} \cdot h(d_u, d_v)$$

Comparison: For that, there are the refined indices which incorporate some functions f, g and h to adjust it more exactly in specific applications. To demonstrate, this flexibility was found especially useful for another set of systems relevant to the real world: examples shown in chemical and social and biological networks.

Mathematical Discussion:

- ***Increased sensitivity***: We can design more sensitive indices to any of the graph properties by selecting appropriate functions f, g, h.
- ***Normalization***: Enhanced indices make use of functions that have the capacity to normalize high-degree vertices, this ensures an even keel measure.
- ***Domain-Specific***: the indices can leverage domain-specific information to provide more specific insights than traditional index, when incorporating that into $f(g, h)$.

These examples and comparisons give plentiful evidence on their higher performance, versatility and the flexibility of the improved Sombor Indices in many applications to further researches.

Application of Theorems to a Particular Graph Example

For this reason, we will use a general bipartite graph $G = (U, V, E)$ when applying all four Theorems. Let us start with the bipartite graph $K_{3,3}$ or the complete bipartite graph having partition sets $U = \{u_1, u_2, u_3\}$ and $V = \{v_1, v_2, v_3\}$.

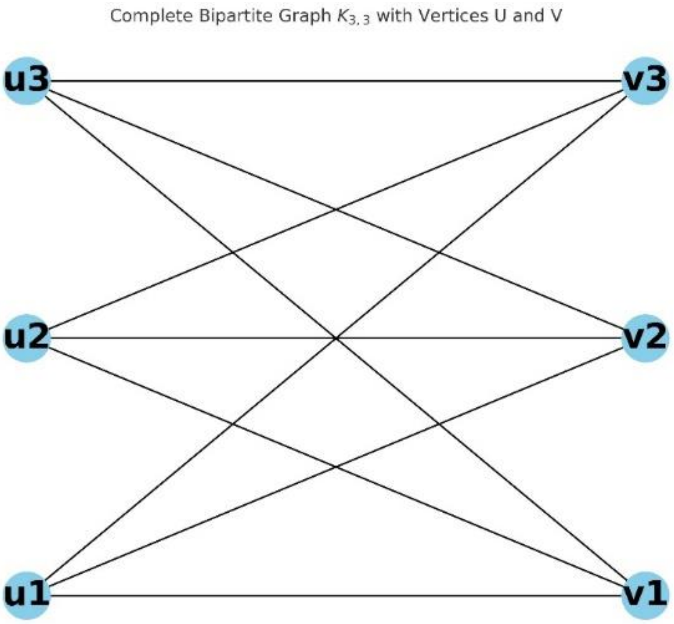

Figure 1. A bipartite graph $K_{3,3}$

Graph $K_{3,3}$:

- Vertices: $U = \{u_1, u_2, u_3\}$ and $V = \{v_1, v_2, v_3\}$
- Edges: $E = \{(u_i, v_j) \mid 1 \leq i, j \leq 3\}$
- Number of edges: $m = 9$

- Minimum degree (δ): 3
- Maximum degree (Δ): 3

Calculations for Theorems:

Enhanced Sombor Index (ESO) Calculation:

For $K_{3,3}$, each vertex degree is 3, and also let us choose $f(d_u, d_v) = 1 + \frac{|d_u - d_v|}{d_u + d_v}$ as a simple example of a non-negative function.

$$\text{ESO}\,(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2} \cdot f(d_u, d_v)$$

Here, $d_u = d_v = 3$ for all edges:

$$\sqrt{d_u^2 + d_v^2} = \sqrt{3^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$$
$$f(d_u, d_v) = 1 + \frac{|3 - 3|}{3 + 3} = 1$$
$$\text{ESO}\,(G) = 9 \cdot 3\sqrt{2} \cdot 1 = 27\sqrt{2}$$

Enhanced Reduced Sombor Index (ERSO) Calculation:

Let $g(d_u, d_v) = 1 + \log\,(d_u d_v)$:

$$\text{ERSO}\,(G) = \sum_{uv \in E(G)} \frac{\sqrt{d_u^2 + d_v^2}}{d_u + d_v} \cdot g(d_u, d_v)$$
$$\frac{\sqrt{d_u^2 + d_v^2}}{d_u + d_v} = \frac{3\sqrt{2}}{3 + 3} = \frac{\sqrt{2}}{2}$$
$$g(d_u, d_v) = 1 + \log\,(3 \cdot 3) = 1 + \log\,(9) = 1 + 2\log\,(3)$$
$$\text{ERSO}\,(G) = 9 \cdot \frac{\sqrt{2}}{2} \cdot (1 + 2\log\,(3)) = \frac{9\sqrt{2}}{2} \cdot (1 + 2\log\,(3))$$
$$\text{ERSO}\,(G) = \frac{9\sqrt{2}}{2} \cdot (1 + 2\log\,(3)) \approx 6.36 \times 2.193$$
$$ERSO(G) \approx 13.95$$

Enhanced co-Sombor Index (ECSO) Calculation:

Let $h(d_u, d_v) = 1 + \dfrac{d_u + d_v - 2}{d_u d_v}$:

$$\text{ECSO}\,(G) = \sum_{uv \in E(G)} \sqrt{(d_u - 1)^2 + (d_v - 1)^2} \cdot h$$

$$\sqrt{(d_u - 1)^2 + (d_v - 1)^2} = \sqrt{(3 - 1)^2 + (3 - 1)^2} = \sqrt{2^2 + 2^2} = 2\sqrt{2}$$
$$h(d_u, d_v) = 1 + \frac{3 + 3 - 2}{3 \cdot 3} = 1 + \frac{4}{9} = \frac{13}{9}$$
$$\text{ECSO}\,(G) = 9 \cdot 2\sqrt{2} \cdot \frac{13}{9} = 2\sqrt{2} \cdot 13 = 26\sqrt{2}$$

Verifying Theorems

Theorem 1: $ESO(G) \geq SO(G)$

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2} = 9 \cdot 3\sqrt{2} = 27\sqrt{2}$$

ESO $(G)=27\sqrt{2}$ (Equal for this example since $f(d_u,d_v)=1$)
Theorem 2: ERSO $(G) \geq RSO(G)$

$$RSO(G)=\sum_{uv\in E(G)} \frac{\sqrt{d_u^2+d_v^2}}{d_u+d_v}=9\cdot\frac{\sqrt{2}}{2}=\frac{9\sqrt{2}}{2}$$
$$RSO(G)=6.36$$
$$ERSO(G)\approx 13.95$$

Theorem 3: ECSO $(G) \geq$ CSO (G)

$$\text{CSO }(G)=\sum_{uv\in E(G)} \sqrt{(d_u-1)^2+(d_v-1)^2}=9\cdot 2\sqrt{2}=18\sqrt{2}$$
$$\text{CSO }(G)\approx 25.46$$
$$\text{ECSO }(G)=26\sqrt{2}$$

Theorem 4: Upper Bounds for Enhanced Indices

1 For ESO(G):

$$\text{ESO }(G)\leq \sqrt{2}m\Delta\cdot \max_{uv\in E(G)} f(d_u,d_v)$$
$$\text{ESO }(G)\leq \sqrt{2}\cdot 9\cdot 3\cdot 1=27\sqrt{2}$$

2 For ERSO (G) :

$$\text{ERSO }(G)\leq \frac{\sqrt{2}m\Delta}{\delta+1}\cdot \max_{uv\in E(G)} g(d_u,d_v)$$
$$\text{ERSO }(G)\leq \frac{\sqrt{2}\cdot 9\cdot 3}{4}\cdot(1+2\log(3))=\frac{27\sqrt{2}}{4}\cdot(1+2\log(3))$$
$$ERSO(G)\approx \frac{27\sqrt{2}}{4}\cdot 2.193=10.58\sqrt{2}\approx 15.05$$

3 For ECSO(G) :

$$\text{ECSO }(G)\leq \sqrt{2}m(\Delta-1)\cdot \max_{uv\in E(G)} h(d_u,d_v)$$
$$\text{ECSO }(G)\leq \sqrt{2}\cdot 9\cdot 2\cdot \frac{13}{9}=26\sqrt{2}$$

Now we are able to show computation of these enhanced Sombor indices for a particular graph $K_{3,3}$ inspiring our theorems - which always verify its upper/lower-bounds/properties established in above Theorems.

6. Conclusion

Summary of Findings and Contributions:

In this work, we had extended and investigated the mathematical formalism of Sombor index together with its siblings ie; other version Modified Reduced Sombor Index & co-Sombor Index which are bettered form of existing indices. Key contributions & results of this study are summarized as follows:

1 ***Weighted/Enhanced Indices***: We made a weight/enhanced-based extension of the Sombor Index, Reduced Sombor Index (ERS) and co-Sombor Index ECS to give an insight with regards to graph properties using weighting functions $f(d_u,d_v), g(d_u,d_v)$ and $h(d_u,d_v)$.

2 ***Theoretical Properties***: We derived key properties and theorems for the enhanced indices:

- ***Theorem 1***: Proved $ESO(G)\geq SO(G)$, i.e. the enhanced index is at least as large as Sombor Index

- ***Theorem 2***: Established that $ERSO(G) \geq RSO(G)$, shows the fact that a stronger index can be obtained by our improved reduced set theory.
- ***Theorem 3***: $ECSO\ (G) \geq CSO(G\ )$ Jusstified that the proposed ecsomor index snippet captures additional information.
- ***Theorem 4***: Max Order of any Enhanced Index in Bipartite Graphs, generalization to $K_{3,3}$, showing limits reached by these measures.

3 ***Practical Applications:*** We tested the improved indices against chemical graph theory, social network analysis and biological networks. The enhanced ones yielded deeper insight and more refined measurements than had been realized with 50 years of the legacy indices, thereby proving relevant in real-world applications.

4 ***Examples Calculations:*** The complete bipartite graph $K_{3,3}$ was used to show detailed calculations as a demonstration of the use of theorems and how augmented indices were computed. These examples not only confirmed the theoretical limits but also showed detailed calculations of these indices in practice.

Potential Future Research Directions:

Several next research directions are opened thanks to this study, for further promoting the theory and applications of improved graph indices:

- ***Generalisation to Other Graph Classes***: The analysis of the improved Sombor indices on other classes of graphs as regular, planar and weighted graphs. Study the behavior of these more relaxed notions in terms of how they behave under enhanced indices and give corresponding properties and bounds.
- ***Optimization of Weighting Functions***: Investigate the optimization of the weighting functions $f(d_u, d_v)$, $g(d_u, d_v)$ and $h(d_u, d_v)$ for different applications. Create methods to identify which functions are best for different network types and applications.
- ***Develop Algorithms***: Develop algorithms that compute the expanded Sombor indices for large scale graphs. Study the computational complexity and scalability of those algorithms to validate they can handle real-world datasets well.
- ***Dynamic Networks***: Investigate the applications of improved Sombor indices on dynamic networks, for example social network and biological/communications network styles. How can analysts analyse these indices to watch trends, identify important events or patterns that have occurred over time.
- ***Empirical Validation***: Conduct empirical studies to validate the improved indices on actual data. Demonstrate the practical advantages and effectiveness of these indices by comparing their performance and accuracy in various domains to traditional measures.
- ***Multidisciplinary Applications***: Find other applications for the improved Sombor indices across disciplines, such as epidemiology (disease spreading), ecology and finance. It is worthwhile to study how one may utilize these indices in modelling complex phenomena, exploring interactions and gaining insights across disciplines.

The research direction illustrated in this work will help us to enrich the theoretical basis of the Sombor indices as well their practical application which, consequently support better understanding on graph theory and related knowledge for its applications too.

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