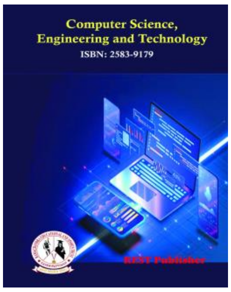
Computer Science, Engineering and Technology

Vol: 2(1), March 2024

REST Publisher; ISSN: 2583-9179 (Online)

Website: https://restpublisher.com/journals/cset/

DOI: https://doi.org/10.46632/cset/2/1/5



Positive Implicative and Associative *Wi*-Ideals of *RLW*-Algebras

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Abstract. *In this paper, we study positive implicative WI-ideal and an associative WI-ideal of RLW-algebra and investigate some of their properties. Also, we prove that every positive implicative WI-ideal is an implicative WI-ideal and hence a WI-ideal, and that every associative WI-ideal is a WI-ideal.*

Keywords-- *W-algebra; LW-algebra; RLW-algebra; RLHW-algebra; Lattice ideal; Ideal; Implicative WI-ideal; Positive Implicative WI-ideal; Associative Implicative WI-ideal.*

Mathematical Subject classification 2010: 03B60, 06B10, 06B20

1. INTRODUCTION

Mordchaj Wajsberg [1] introduced the concept of W-algebras in 1935 and studied by Font, Rodriguez and Torrens[2]. Residuated lattices were announced by Ward and Dilworth [3]. Ibrahim and Shajitha Begum [4] introduced the notions of LW-algebras and also investigated their properties with suitable llustrations. The authors [5] introduced the notion of anti-fuzzy Wajsberg implicative ideal (AFWI-ideal) of RLW-algebras.

In this paper, we consider positive implicative WI-ideal of RLW-algebra and investigate some related properties. Also, we prove that every positive implicative WI-ideal is an implicative WI-ideal and hence a WI-ideal, and that every associative WI-ideal is a WI-ideal.

2. PRELIMINARIES

In this section, we recall some basic definitions and properties which are helpful to develop our main results.

Definition 2.1[3]. A residuated lattice $(\wp, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ satisfied the following conditions for all $\mathfrak{o}, \mathfrak{p}, \mathfrak{q} \in \wp$,

(i) $(\wp, \vee, \wedge, 0, 1)$ is a bounded lattice
(ii) $(\wp, \otimes, 1)$ is commutative monoid
(iii) $\mathfrak{o} \otimes \mathfrak{p} \leq \mathfrak{q}$ if and only if $\mathfrak{o} \leq \mathfrak{p} \rightarrow \mathfrak{q}$.

Definition 2.2[2]. A W-algebra $(\wp, \rightarrow, *, 1)$satisfied the following axioms for all $\mathfrak{o}, \mathfrak{p}, \mathfrak{q} \in \wp$,

(i) $\mathfrak{o} \rightarrow \mathfrak{o} = 1$
(ii) If $(\mathfrak{o} \rightarrow \mathfrak{p}) = (\mathfrak{p} \rightarrow \mathfrak{o}) = 1$ then $\mathfrak{o} = \mathfrak{p}$
(iii) $\mathfrak{o} \rightarrow 1 = 1$
(iv) $(\mathfrak{o} \rightarrow (\mathfrak{p} \rightarrow \mathfrak{o})) = 1$
(v) If $(\mathfrak{o} \rightarrow \mathfrak{p}) = (\mathfrak{p} \rightarrow \mathfrak{q}) = 1$ then $\mathfrak{o} \rightarrow \mathfrak{q} = 1$
(vi) $(\mathfrak{o} \rightarrow \mathfrak{p}) \rightarrow \big((\mathfrak{q} \rightarrow \mathfrak{o}) \rightarrow (\mathfrak{q} \rightarrow \mathfrak{p})\big) = 1$
(vii) $\mathfrak{o} \rightarrow (\mathfrak{p} \rightarrow \mathfrak{q}) = \mathfrak{p} \rightarrow (\mathfrak{o} \rightarrow \mathfrak{q})$

(viii) $\mathfrak{o} \to 0 = \mathfrak{o} \to 1^* = \mathfrak{o}^*$
(ix) $(\mathfrak{o}^*)^* = \mathfrak{o}$
(x) $(\mathfrak{o}^* \to \mathfrak{p}^*) = \mathfrak{p} \to \mathfrak{o}$.

Proposition 2.3[3]. Let $(\wp, \vee, \wedge, \otimes, \to, 0, 1)$ be a residuated lattice. Then the following are satisfied for all $\mathfrak{o}, \mathfrak{p}, \mathfrak{q} \in \wp$,

(i) $(\mathfrak{o} \otimes \mathfrak{p}) \to \mathfrak{q} = \mathfrak{o} \to (\mathfrak{p} \to \mathfrak{q})$
(ii) $(\mathfrak{o} \otimes \mathfrak{p}) \otimes \mathfrak{q} = \mathfrak{o} \otimes (\mathfrak{p} \otimes \mathfrak{q})$
(iii) $\mathfrak{o} \otimes \mathfrak{p} = \mathfrak{p} \otimes \mathfrak{o}$

Definition 2.4[2]. Let $(\wp, \vee, \wedge, *, \to, 1)$ be a LW-algebra. If a binary operation " $\otimes$ " on $\wp$ satisfied $\mathfrak{o} \otimes \mathfrak{p} = (\mathfrak{o} \to \mathfrak{p}^*)^*$ for all $\mathfrak{o}, \mathfrak{p} \in \wp$. Then $(\wp, \vee, \wedge, \otimes, \to, *, 0, 1)$is called a RLW-algebra.
Definition 2.5[6]. The RLW-algebra $\wp$ is called a $RLHW$-algebra if it satisfied $\mathfrak{o} \vee \mathfrak{p} \vee ((\mathfrak{o} \wedge \mathfrak{p}) \to \mathfrak{q}) = 1$for all $\mathfrak{o}, \mathfrak{p}, \mathfrak{q} \in \wp$.

In a $RLHW$-algebra $\wp$, the following are hold,
(i) $\mathfrak{o} \otimes \mathfrak{p} \in \wp$
(ii) $\mathfrak{o} \otimes (\mathfrak{o} \otimes \mathfrak{p}) = (\mathfrak{o} \otimes \mathfrak{p})$; $\mathfrak{o} \to (\mathfrak{o} \to \mathfrak{p}) = (\mathfrak{o} \to \mathfrak{p})$
(iii) $\mathfrak{o} \otimes (\mathfrak{p} \otimes \mathfrak{q}) = (\mathfrak{o} \otimes \mathfrak{p}) \otimes (\mathfrak{o} \otimes \mathfrak{q})$; $\mathfrak{o} \to (\mathfrak{p} \to \mathfrak{q}) = (\mathfrak{o} \to \mathfrak{p}) \to (\mathfrak{o} \to \mathfrak{q})$
Definition 2.6[2]. Let $\wp$ be a lattice. An ideal I of $\wp$ is a nonempty subset of $\wp$ is called a lattice ideal, if it satisfied the following axioms for all $\mathfrak{o}, \mathfrak{p} \in \wp$,

(i) $\mathfrak{o} \in I, \mathfrak{p} \in L$ and $\mathfrak{p} \leq \mathfrak{o}$ imply $\mathfrak{p} \in I$
(ii) $\mathfrak{o}, \mathfrak{p} \in I$implies$\mathfrak{o} \vee \mathfrak{p} \in I$.

Definition 2.7[4]. A non-empty subset I of a W-algebra $\wp$ is an ideal, if it satisfied the following axioms for all $\mathfrak{o}, \mathfrak{p} \in \wp$,

(i) $0 \in I$
(ii) $\mathfrak{o} \in I$ and $\mathfrak{p} \leq \mathfrak{o}$ imply $\mathfrak{p} \in I$.

3. MAIN RESULTS

3.1 Positive Implicative and Associative *WI*-ideals of RLW- algebras.

In this section, we consider positive implicative and associative *WI*-ideals of RLW-algebra and explore some of its properties.

Definition 3.1.1. A non-empty subset I of a RLW-algebra $\wp$ is called a positive implicative *WI*-ideal of $\wp$ if it satisfies the following,

(i) $0 \in I$;
(ii) $(\mathfrak{p} \otimes (\mathfrak{q} \otimes \mathfrak{p})) \otimes \mathfrak{o}) \in I$ and $\mathfrak{o} \in I$ imply $\mathfrak{p} \in I$ for all $\mathfrak{o}, \mathfrak{p}, \mathfrak{q} \in \wp$;
(iii) $((\mathfrak{p} \to (\mathfrak{q} \to \mathfrak{p})^*)^* \to \mathfrak{o})^* \in I$ and $\mathfrak{o} \in I$ imply $\mathfrak{p} \in I$ for all $\mathfrak{o}, \mathfrak{p}, \mathfrak{q} \in \wp$.

Example 3.1.2. Consider a set $\wp$=$\{0, \mathcal{u}, \mathcal{v}, \mathcal{w}, \mathcal{s}, \mathcal{t}, 1\}$. Define a partial ordering "$\leq$" on $\wp$, such that $0 \leq \mathcal{u} \leq \mathcal{v} \leq \mathcal{w} \leq \mathcal{s} \leq \mathcal{t} \leq 1$ with a binary operations " $\otimes$ "and " $\to$ "and a quasi complement " $*$ "on $\wp$ as in following tables 3.1.1 and 3.1.2.

Table 1. Complement

$\mathfrak{o}$	$\mathfrak{o}^*$
0	1
u	s
v	s
w	v
s	v
t	0
1	0

Table 2. Implication

$\to$	0	u	v	w	s	t	1
0	1	1	1	1	1	1	1
u	s	1	1	s	s	1	1
v	s	t	1	s	s	1	1
w	v	v	v	1	1	1	1
s	v	v	v	t	1	1	1
t	0	v	v	s	s	1	1
1	0	u	v	w	s	t	1

Define ∨ and ∧ operations on $\wp$ as follows:

$(\mathfrak{o} \vee \mathfrak{p}) = (\mathfrak{o} \to \mathfrak{p}) \to \mathfrak{p}$,

$(\mathfrak{o} \wedge \mathfrak{p}) = (\mathfrak{o}^* \to \mathfrak{p}^*) \to \mathfrak{p}^*)^*$; $\mathfrak{o} \otimes y = (\mathfrak{o} \to \mathfrak{p}^*)^*$ for all $\mathfrak{o}, \mathfrak{p} \in \wp$.

Then, $\wp$ is a RLW-algebra. It is easy to verify that, $I_1 = \{0, u, s\}$ is an positive implicative WI-ideal of $\wp$. But $I_2 = \{v, w, s\}$ is not a positive implicative WI-ideal of $\wp$. Since, $\big((w \otimes (s \otimes w)) \otimes v\big) = 0 \notin I_2$.

Proposition 3.1.3. Let I be a non-empty subset of $\wp$. If I is a positive implicative WI-ideal of $\wp$, then I is a WI-ideal of $\wp$.

Proof. Let I be a positive implicative WI-ideal of $\wp$ then from the definition 3.1.1 we have $0 \in I$ and

replace $\mathfrak{o} = \mathfrak{p}$ and $\mathfrak{q} = \mathfrak{o}$ for all $\mathfrak{o}, \mathfrak{p}, \mathfrak{q} \in \wp$ in (ii) of the definition 3.1.1, $(\big((\mathfrak{o} \otimes (\mathfrak{o} \otimes \mathfrak{o})) \otimes \mathfrak{p}\big)) \in I$, $(((\mathfrak{o} \to (\mathfrak{o} \to \mathfrak{o})^*) \to \mathfrak{p})^* \in I$ and$\mathfrak{p} \in I$ for all $\mathfrak{o}, \mathfrak{p}, \mathfrak{q} \in \wp$

$(\mathfrak{o} \otimes 0) \otimes \mathfrak{p} \in I$, $(((\mathfrak{o} \to 0)^* \to \mathfrak{p})) \in I$ and $\mathfrak{p} \in I$ imply$\mathfrak{o} \in I$ for all $\mathfrak{o}, \mathfrak{p}, \mathfrak{q} \in \wp$

$\mathfrak{o} \otimes \mathfrak{p} \in I$, $(\mathfrak{o} \to \mathfrak{p})^* \in I$ and $\mathfrak{p} \in I$ imply $\mathfrak{o} \in I$ for all $\mathfrak{o}, \mathfrak{p}, \mathfrak{q} \in \wp$

Thus, I is a WI-ideal of $\wp$.

Proposition 3.1.4. Let I be a WI-ideal $\wp$. Then I is a positive implicative WI-ideal $\wp$ if and only if $\mathfrak{o} \otimes (\mathfrak{p} \otimes \mathfrak{o}) \in I$, $(\mathfrak{o} \to (\mathfrak{p} \to \mathfrak{o})^*)^* \in I$ implies $\mathfrak{o} \in I$ for all $\mathfrak{o}, \mathfrak{p} \in \wp$.

Proof. Let I be a positive implicative WI-ideal of $\wp$ and let $\mathfrak{o} = 0, \mathfrak{p} = \mathfrak{o}, \mathfrak{q} = \mathfrak{p}$ in $\mathfrak{p} \otimes (\mathfrak{q} \otimes \mathfrak{p}) \otimes \mathfrak{o} \in I$, $(((\mathfrak{p} \to (\mathfrak{q} \to \mathfrak{p})^*)^* \to \mathfrak{o})^*) \in I$ and $\mathfrak{o} \in I$ imply $\mathfrak{p} \in I$ then, we have $(\mathfrak{o} \otimes (\mathfrak{p} \otimes \mathfrak{o})) \otimes 0 \in I$, $(((\mathfrak{o} \to (\mathfrak{p} \to \mathfrak{o})^*)^* \to 0)^* \in I$ and $0 \in I$ imply $\mathfrak{o} \in I$, which implies that, $\mathfrak{o} \otimes (\mathfrak{p} \otimes \mathfrak{o}) \in I$, $((\mathfrak{o} \to (\mathfrak{p} \to \mathfrak{o})^*)^* \in I$ implies $\mathfrak{o} \in I$.

Conversely, since I is a WI-ideal $\wp$, $\mathfrak{p} \otimes (\mathfrak{q} \otimes \mathfrak{p}) \in I$, $((\mathfrak{p} \to (\mathfrak{q} \to \mathfrak{p})^*)^* \in I$.

Thus, we have $\mathfrak{p} \in I$.

Proposition 3.1.5. Let I be a non-empty subset of RLW-algebra $\wp$. If I is a positive implicative WI-ideal of $\wp$, then it is an implicative WI-ideal of $\wp$.

Proof. Let I be a positive implicative WI-ideal of $\wp$.

We need to prove: I is an implicative WI-ideal of $\wp$.

Let $(\mathfrak{o} \otimes \mathfrak{p}) \otimes \mathfrak{q}, ((\mathfrak{o} \to \mathfrak{p})^* \to \mathfrak{q})^* \in I$ an $\mathfrak{p} \otimes \mathfrak{q}, (\mathfrak{p} \to \mathfrak{q})^* \in I$.

It is enough to show that $\mathfrak{o} \otimes \mathfrak{q}, (\mathfrak{o} \to \mathfrak{q})^* \in I$

Here, $(\mathfrak{o} \otimes \mathfrak{p}) \otimes \mathfrak{q} = \mathfrak{o} \otimes (\mathfrak{p} \otimes \mathfrak{q})$ [From (ii) of proposition 2.3]

$= \mathfrak{o} \otimes (\mathfrak{q} \otimes \mathfrak{p})$ [From (iii) of proposition 2.3]

$= (\mathfrak{o} \otimes \mathfrak{q}) \otimes \mathfrak{p})$ [From (ii) of proposition 2.3]

$((\mathfrak{o} \to \mathfrak{p})^* \to \mathfrak{q})^* = (\mathfrak{q}^* \to (\mathfrak{o} \to \mathfrak{p}))^*$

$= (\mathfrak{o} \to (\mathfrak{q}^* \to \mathfrak{p}))^*$ [From (vii) of proposition 2.2]

$= (\mathfrak{o} \to (\mathfrak{p}^* \to \mathfrak{q}))^*$ [From (x) of proposition 2.2]

$= (\mathfrak{p}^* \to (\mathfrak{o} \to \mathfrak{q}))^*$ [From (vii) of proposition 2.2]

$= ((\mathfrak{o} \to \mathfrak{q})^* \to \mathfrak{p})^*$ [From (x) of proposition 2.2]

Therefore, $((\mathfrak{o} \to \mathfrak{p})^* \to \mathfrak{q})^* = ((\mathfrak{o} \to \mathfrak{q})^* \to \mathfrak{p})^*$

We prove that, $(\mathfrak{o} \otimes \mathfrak{q}) \to \mathfrak{p} \leq ((\mathfrak{p} \to \mathfrak{q}) \to ((\mathfrak{o} \otimes \mathfrak{q}) \to \mathfrak{q})$

Then $((\mathfrak{p} \to \mathfrak{q}) \to ((\mathfrak{o} \otimes \mathfrak{q}) \otimes \mathfrak{p})$

$(((\mathfrak{o} \otimes \mathfrak{q}) \otimes \mathfrak{q}) \otimes (\mathfrak{p} \otimes \mathfrak{q}) \leq (\mathfrak{o} \otimes \mathfrak{q}) \otimes \mathfrak{p}y$ and

$((\mathfrak{o} \to \mathfrak{q})^* \to \mathfrak{p} \leq ((\mathfrak{p} \to \mathfrak{q}) \to ((\mathfrak{o} \to \mathfrak{q})^* \to \mathfrak{q})$ then $(((\mathfrak{p} \to \mathfrak{q}) \to ((\mathfrak{o} \to \mathfrak{q})^* \to \mathfrak{q}))^* \leq ((\mathfrak{o} \to \mathfrak{q})^* \to \mathfrak{p})^*$

Since, $\left((\mathfrak{o} \otimes \mathfrak{q}) \otimes \mathfrak{p}\right), \mathfrak{p} \otimes \mathfrak{q}, ((\mathfrak{o} \to \mathfrak{q})^* \to \mathfrak{p})^*, (\mathfrak{p} \to \mathfrak{q})^* \in I$

We have $\left((\mathfrak{o} \otimes \mathfrak{q}) \otimes \mathfrak{q}\right), ((\mathfrak{o} \to \mathfrak{q})^* \to \mathfrak{q})^* \in I$

Also, $\left((\mathfrak{o} \otimes \mathfrak{q}) \otimes \mathfrak{q}\right) = ((\mathfrak{o} \otimes \mathfrak{q}) \otimes 0) \otimes \mathfrak{q}$

$= ((\mathfrak{o} \otimes \mathfrak{q}) \otimes (0 \otimes \mathfrak{q})) \otimes \mathfrak{q})$

$= (\mathfrak{o} \otimes \mathfrak{q}) \otimes (((\mathfrak{o} \otimes \mathfrak{o}) \otimes \mathfrak{q}) \otimes \mathfrak{q})$

$= ((\mathfrak{o} \otimes \mathfrak{q}) \otimes \left((\mathfrak{o} \otimes \mathfrak{q}) \otimes \mathfrak{o}\right)) \otimes \mathfrak{q})$ [From (ii) of definition 2.5]

$((\mathfrak{o} \to \mathfrak{q})^* \to \mathfrak{q})^* = (((\mathfrak{o} \to \mathfrak{q})^* \to 0)^* \to \mathfrak{q})^*$

$= (((\mathfrak{o} \to \mathfrak{q})^* \to (0 \to \mathfrak{q})^*)^* \to \mathfrak{q})^*$

$= ((\mathfrak{o} \to \mathfrak{q})^* \to ((\mathfrak{o} \to \mathfrak{o})^* \to \mathfrak{q})^*)^* \to \mathfrak{q})^*$ [From (i) of proposition 2.2]

$= (((\mathfrak{o} \to \mathfrak{q})^* \to ((\mathfrak{o} \to \mathfrak{q})^* \to \mathfrak{o})^*)^* \to \mathfrak{q})^*$

From (iii) of definition 2.1, we have

$\left(\mathfrak{o} \otimes \left(\mathfrak{o} \otimes (\mathfrak{o} \otimes \mathfrak{q})\right)\right) \otimes \mathfrak{q} = ((\mathfrak{o} \otimes \mathfrak{q}) \otimes \left(\mathfrak{o} \otimes (\mathfrak{o} \otimes \mathfrak{q})\right))$

$((\mathfrak{o} \to (\mathfrak{o} \to (\mathfrak{o} \to \mathfrak{q})^*)^*)^* \to \mathfrak{q})^* = ((\mathfrak{o} \to \mathfrak{q})^* \to (\mathfrak{o} \to (\mathfrak{o} \to \mathfrak{q})^*)^*)^*$

Thus , we have $\mathfrak{o} \otimes \mathfrak{q}$, $(\mathfrak{o} \to \mathfrak{q})^* \in I$.

Proposition 3.1.6. Let I be a non-empty subset of $RLHW$-algebra $\wp$. If I is an implicative WI-ideal of $\wp$, then I is a positive implicative WI-ideal of $\wp$.

Proof. Let I be an implicative WI-ideal of $RLHW$-algebra $\wp$,

Then, we have $= \mathfrak{p} \otimes (\mathfrak{q} \otimes \mathfrak{p}), (\mathfrak{p} \to (\mathfrak{q} \to \mathfrak{p})^*)^* \in I$

Thus, we get $\mathfrak{p} \otimes (\mathfrak{q} \otimes \mathfrak{p}) = \mathfrak{p} \otimes (\mathfrak{p} \otimes \mathfrak{q})$

$= \mathfrak{p} \otimes \mathfrak{q}$ [From (ii) of definition 2.5]

$= 0 \in I$ and

$(\mathfrak{p} \to (\mathfrak{q} \to \mathfrak{p})^*)^* = ((\mathfrak{q} \to \mathfrak{p}) \to \mathfrak{p}^*)^* = ((\mathfrak{p}^* \to \mathfrak{q}^*) \to \mathfrak{p}^*)^*$.

Since, $\wp$ is a $RLHW$-algebra, we get $\mathfrak{p} = \mathfrak{p} \otimes (\mathfrak{q} \otimes \mathfrak{p})$, $\mathfrak{p} = (\mathfrak{p} \to (\mathfrak{q} \to \mathfrak{p})^*)^* \in I$.

Proposition 3.1.7. Let M and N be two WI-ideals of RLW-algebra $\wp$ with $M \subseteq N$. If M is a positive implicative WI-ideal of $\wp$ then so is N.

Proof. Let $\mathfrak{o} \otimes (\mathfrak{p} \otimes \mathfrak{o}) \in N$.

Take $r = \mathfrak{o} \otimes (\mathfrak{p} \otimes \mathfrak{o})$, $(\mathfrak{o} \to (\mathfrak{p} \to \mathfrak{o})^*)^*$, $X = \mathfrak{o} \otimes r, (\mathfrak{o} \to r)^*$ and $Y = \mathfrak{o}$.

Then, $Y \otimes \mathrm{X} = \mathfrak{o} \otimes (\mathfrak{o} \otimes \mathrm{r})$

$= \mathfrak{o} \otimes (\mathfrak{o} \otimes (\mathfrak{o} \otimes (\mathfrak{p} \otimes \mathfrak{o})))$

$= \mathfrak{o} \otimes (\mathfrak{o} \otimes (\mathfrak{o} \otimes (\mathfrak{o} \otimes \mathfrak{p})))$ [From (iii) of definition 2.3]

$= \mathfrak{o} \otimes (\mathfrak{o} \otimes (\mathfrak{o} \otimes \mathfrak{p}))$ [From (ii) of definition 2.5]

$= \mathfrak{o} \otimes (\mathfrak{o} \otimes \mathfrak{p})$ [From (ii) of definition 2.5]

$= \mathfrak{o} \otimes (\mathfrak{p} \otimes \mathfrak{o})$ [From (iii) of definition 2.3]

$= r$ and

$(Y \to X)^* = (\mathfrak{o} \to (\mathfrak{o} \to r)^*)^*$

$= (\mathfrak{o} \to (\mathfrak{o} \to (\mathfrak{o} \to (\mathfrak{p} \to \mathfrak{o})^*)^*)^*)^*$

$= ((\mathfrak{o} \to (\mathfrak{p} \to \mathfrak{o})^*)^*)^* = r^*$

Therefore, $Y \otimes \mathrm{X} = r$, $(Y \to X)^* = r^*$

So, $(X \otimes (\mathrm{Y} \otimes \mathrm{X}) = X \otimes r$

$= (\mathfrak{o} \otimes r) \otimes r$

$= r \otimes (\mathfrak{o} \otimes r)$ [From (iii) of definition 2.3]

$= r \otimes (r \otimes \mathfrak{o})$ [From (iii) of definition 2.3]

$= (r \otimes \mathfrak{o})$ [From (ii) of definition 2.5]

$= (\mathfrak{o} \otimes (\mathfrak{p} \otimes \mathfrak{o})) \otimes \mathfrak{o}$

$= (\mathfrak{o} \otimes (\mathfrak{o} \otimes \mathfrak{p})) \otimes \mathfrak{o}$ [From (iii) of definition 2.3]

$= (\mathfrak{o} \otimes \mathfrak{p}) \otimes \mathfrak{o}$ [From (ii) of definition 2.5]

$= \mathfrak{o} \otimes (\mathfrak{o} \otimes \mathfrak{p})$ [From (iii) of definition 2.3]

$= \mathfrak{o} \otimes y \in M$ and

$(X \to (Y \to X)^*)^* = ((\mathfrak{o} \to r)^*) \to r^*)^*$

$= (r \to (\mathfrak{o} \to r))^*$

$= (\mathfrak{o} \to (r \to r))^*$

$(X \to (Y \to X)^*)^* = 0 \in M$

So $\mathfrak{o} \in M$ by M is a positive implicative WI-ideal of $\wp$.

Since $M \subseteq N$, $\mathfrak{o} \otimes r$, $(\mathfrak{o} \to r)^* = X \in N$ implies that $\mathfrak{o} \in N$. Thus, N is a positive WI-ideal of $\wp$.

3.2. Associative WI-ideals of RLW-algebras.

In this section, we introduce the concept of associative WI-ideal of RLW-algebra and we find some of its properties with illustrations.

Definition 3.2.1. A subset of $\wp$ is said to be an associative WI-ideal of $\wp$ with respect to $\mathfrak{o}$, where $\mathfrak{o}$ is fixed element of $\wp$, if it satisfies the following axioms for all $\mathfrak{o}, y \in \wp$ and $\mathfrak{o} \neq 1$,

(i) $0 \in I$
(ii) $\mathfrak{p} \otimes \mathfrak{o} \in I$ and $((\mathfrak{q} \otimes \mathfrak{p}) \otimes \mathfrak{o}) \in I$ imply $\mathfrak{q} \in I$
(iii) $(\mathfrak{p} \to \mathfrak{o})^* \in I$ and $((\mathfrak{q} \to \mathfrak{p})^* \to \mathfrak{o}^*)$ imply$\mathfrak{q} \in I$.

Example 3.2.2. Consider a set $\wp$=$\{0, p, q, r, s, t, 1\}$. Define a partial ordering "$\leq$" on $\wp$, such that $0 \leq a \leq b \leq c \leq d \leq 1$ with a binary operations" $\otimes$ "and " $\to$ "and a quasi complement " $*$ "on $\wp$ as in following tables 3.1.3 and 3.1.4.

Table 3. Complement

$\mathfrak{o}$	$\mathfrak{o}^*$
0	1
p	r
q	q
r	p
1	0

Table 4. Implication

$\to$	0	p	q	r	1
0	1	1	1	1	1
p	r	1	1	1	1
q	q	r	1	1	1
r	p	q	1	1	1
1	0	p	q	r	1

Define $\vee$ and $\wedge$ operations on $\wp$ as follows:

$(\mathfrak{o} \vee \mathfrak{p}) = (\mathfrak{o} \to \mathfrak{p}) \to \mathfrak{p}$,

$(\mathfrak{o} \wedge \mathfrak{p}) = (\mathfrak{o}^* \to \mathfrak{p}) \to \mathfrak{p}^*)^*$; $\mathfrak{o} \otimes \mathfrak{p} = (\mathfrak{o} \to \mathfrak{p}^*)^*$ for all $\mathfrak{o}, \mathfrak{p} \in \wp$.

Then, $\wp$ is a RLW-algebra. It is easy to verify that, $I_2 = \{0, q, r\}$ is an associative WI-ideals of $\wp$.

Proposition 3.2.3. Every associative WI-ideal with respect to $\mathfrak{o}$ contains $\mathfrak{o}$ itself.

Proof. Let I be an associative WI-ideal of $\wp$.

If $\mathfrak{o} = 0$ then $\mathfrak{p} \otimes 0$, $(\mathfrak{p} \to 0)^* \in I$ and $(\mathfrak{q} \otimes y) \otimes 0$, $((\mathfrak{q} \to \mathfrak{p})^* \to 0)^* \in I$ imply $\mathfrak{q} \in I$.

So $\mathfrak{p} \in I$ and $\mathfrak{q} \otimes \mathfrak{p}$, $(\mathfrak{q} \to \mathfrak{p})^* \in I$ imply $\mathfrak{q} \in I$.

Hence, we have I is a WI-ideal of $\wp$ that contain 0. If $\mathfrak{o} = 1$ then $I = A$.

If $\mathfrak{o} \neq 0, 1$, take $\mathfrak{p} = 0$ and $\mathfrak{q} = \mathfrak{o}$ then $(\mathfrak{o} \otimes 0) \otimes \mathfrak{o} = (0 \to \mathfrak{q})^* = 1^* = 0 \in I$,

$((\mathfrak{o} \to 0)^* \to \mathfrak{o})^* = (\mathfrak{o} \to \mathfrak{o})^* = 0 \in I$ and $0 \otimes \mathfrak{o}$, $(0 \to \mathfrak{o})^* = 0 \in I$ imply $\mathfrak{o} \in I$.

Proposition 3.2.4. Every associative WI-ideal is a WI-ideal of RLW-algebra $\wp$.

Proof. If $\mathfrak{p} \in I$ and $\mathfrak{o} \otimes \mathfrak{p}$,$(\mathfrak{o} \to \mathfrak{p})^* \in I$ then $\mathfrak{p} \otimes 0$, $(\mathfrak{p} \to 0)^* \in I$ and

$(\mathfrak{o} \otimes \mathfrak{p}) \otimes 0$, $((\mathfrak{o} \to \mathfrak{p})^* \to 0)^* \in I$. Since I is an associative WI-ideal of $\wp$ then $\mathfrak{o} \in I$.

Proposition 3.2.5. Let I be a WI-ideal of $\wp$. I is an associative WI-ideal if and only if $((\mathfrak{q} \otimes \mathfrak{p}) \otimes \mathfrak{o})$, $(((\mathfrak{q} \to \mathfrak{p})^* \to \mathfrak{o})^*)$ implies $\mathfrak{q} \otimes (\mathfrak{p} \otimes \mathfrak{o})$, $((z \to (\mathfrak{p} \to \mathfrak{o})^*)^*) \in I$.

Proof. If $(\mathfrak{q} \otimes \mathfrak{p}) \otimes \mathfrak{o}$, $(((\mathfrak{q} \to \mathfrak{p})^* \to \mathfrak{o})^*)$ and $\mathfrak{p} \otimes \mathfrak{o}$, $(\mathfrak{p} \to \mathfrak{o})^* \in I$ then

$\mathfrak{q} \otimes (\mathfrak{p} \otimes \mathfrak{o})$, $((\mathfrak{q} \to (\mathfrak{p} \to \mathfrak{o})^*)^*)$ and $\mathfrak{p} \otimes \mathfrak{o}$, $(\mathfrak{p} \to \mathfrak{o})^* \in I$

Since I is a WI-ideal of $\wp$, then $\mathfrak{q} \in I$.

Conversely, let $(\mathfrak{q} \otimes \mathfrak{p}) \otimes \mathfrak{o}, (((\mathfrak{q} \to \mathfrak{p})^* \to \mathfrak{o})^*) \in I$ then

$$\left(\left((\mathfrak{q} \otimes (\mathfrak{p} \otimes \mathfrak{o})) \otimes (\mathfrak{q} \otimes \mathfrak{p})\right) \otimes \mathfrak{o}\right) = (((\mathfrak{q} \otimes \mathfrak{p}) \otimes (\mathfrak{q} \otimes \mathfrak{o})) \otimes (\mathfrak{q} \otimes \mathfrak{p})) \otimes = \mathfrak{o} \otimes 0 = 0 \in I$$

Hence, $\left(\left((\mathfrak{q} \otimes (\mathfrak{p} \otimes \mathfrak{o})) \otimes (\mathfrak{q} \otimes \mathfrak{p})\right) \otimes \mathfrak{o}\right) \in I$ (3.2.1)

Equation (3.2.1) comes from$\mathfrak{q} \otimes \mathfrak{p} \leq (\mathfrak{p} \otimes \mathfrak{o}) \otimes (\mathfrak{q} \otimes \mathfrak{o})$

Which implies $(\mathfrak{q} \otimes \mathfrak{o}) \otimes (\mathfrak{p} \otimes \mathfrak{o}) \leq \mathfrak{q} \otimes \mathfrak{p}$ and

$$((((\mathfrak{q} \to (\mathfrak{p} \to \mathfrak{o})^*)^* \to (\mathfrak{q} \to \mathfrak{p})^*)^* \to \mathfrak{o})^*) = ((((\mathfrak{q} \to (\mathfrak{p} \to \mathfrak{o})^*)^* \to \mathfrak{o})(\mathfrak{q} \to \mathfrak{p})^*)^*)$$

$$= ((((\mathfrak{q} \to \mathfrak{o})^* \to (\mathfrak{p} \to \mathfrak{o})^*)^* \to (\mathfrak{q} \to \mathfrak{p})^*)^*) = 1^* = 0 \in I$$

Hence, $((((\mathfrak{q} \to (\mathfrak{p} \to \mathfrak{o})^*)^* \to (\mathfrak{q} \to \mathfrak{p})^*)^* \to \mathfrak{o})^*) \in I$ (3.2.2)

Equation (3.2.2) comes from $(\mathfrak{q} \to \mathfrak{p}) \leq (\mathfrak{p} \to \mathfrak{o}) \to (\mathfrak{q} \to \mathfrak{o})$

Which implies $(((\mathfrak{q} \to \mathfrak{o})^* \to (\mathfrak{p} \to \mathfrak{o})^*)^*) \leq ((\mathfrak{q} \to \mathfrak{p})^*)$.

From our assumption that,$\mathfrak{q} \otimes (\mathfrak{p} \otimes \mathfrak{o}), ((\mathfrak{q} \to (\mathfrak{p} \to \mathfrak{o})^*)^*) \in I$ and I is an associative WI-ideal.

Thus, we have $\mathfrak{q} \otimes (\mathfrak{p} \otimes \mathfrak{o}), ((\mathfrak{q} \to (\mathfrak{p} \to \mathfrak{o})^*)^*) \in I$

Proposition 3.2.6. Let I be a *WI*-ideal of $\wp$. I is an associative *WI*-ideal if and only if $(\mathfrak{p} \otimes \mathfrak{o}) \otimes \mathfrak{o}, ((y \to \mathfrak{o})^* \to \mathfrak{o})^* \in I$ implies $\mathfrak{p} \in I$.

Proof. If $(\mathfrak{p} \otimes \mathfrak{o}) \otimes \mathfrak{o}, ((\mathfrak{p} \to \mathfrak{o})^* \to \mathfrak{o})^* \in I$ then $\mathfrak{p} \otimes (\mathfrak{o} \otimes \mathfrak{o}), ((\mathfrak{p} \to (\mathfrak{o} \to \mathfrak{o})^*)^* \in I$.

So, $\mathfrak{p} \otimes 0 = 0, (\mathfrak{p} \to 0)^* = \mathfrak{p} \in I$

Conversely, $\left(\left((\mathfrak{q} \otimes (\mathfrak{p} \otimes \mathfrak{o})) \otimes \mathfrak{o}\right) \otimes \mathfrak{o}\right) \otimes ((\mathfrak{q} \otimes \mathfrak{p}) \otimes \mathfrak{o}))$

$$= (((\mathfrak{q} \otimes \mathfrak{p}) \otimes (\mathfrak{q} \otimes \mathfrak{o})) \otimes \mathfrak{o}) \otimes \mathfrak{o}) \otimes (\mathfrak{o} \otimes (\mathfrak{q} \otimes y))) \quad (3.2.3)$$

$$= \left(((0 \otimes 0) \otimes \mathfrak{o}) \otimes \mathfrak{o}\right) \otimes ((\mathfrak{o} \otimes \mathfrak{q}) \otimes (\mathfrak{o} \otimes y))$$

$$= \left(((0 \otimes \mathfrak{o}) \otimes \mathfrak{o}) \otimes \mathfrak{o}\right) \otimes (0 \otimes 0))$$

$$= (\left((\mathfrak{o} \otimes (0 \otimes \mathfrak{o})) \otimes \mathfrak{o}\right) \otimes (0 \otimes 0))$$

$$= (((0 \otimes \mathfrak{o}) \otimes \mathfrak{o}) \otimes 0)$$

$$= ((\mathfrak{o} \otimes (0 \otimes \mathfrak{o})) \otimes 0)$$

$$= ((0 \otimes \mathfrak{o}) \otimes 0)$$

$$= (0 \otimes (0 \otimes \mathfrak{o})$$

$= 0 \otimes \mathfrak{o} = 0 \in I$ and

$$((((\mathfrak{q} \to (\mathfrak{p} \to \mathfrak{o})^*)^* \to \mathfrak{o})^* \to \mathfrak{o})^* \to ((\mathfrak{q} \to \mathfrak{p})^* \to \mathfrak{o})^*)^*$$

$$= ((((\mathfrak{q} \to (\mathfrak{p} \to \mathfrak{o})^*)^* \to \mathfrak{o})^* \to \mathfrak{o})^* \to ((\mathfrak{q} \to \mathfrak{p})^* \to \mathfrak{o})^*)^* \to 0)^*$$

$$= ((((\mathfrak{q} \to (\mathfrak{p} \to \mathfrak{o})^*)^* \to \mathfrak{o})^* \to \mathfrak{o})^* \to (((\mathfrak{q} \to \mathfrak{o})^* \to (\mathfrak{p} \to \mathfrak{o})^*)^* \to (\mathfrak{q} \to \mathfrak{p})^*)^*)^* \quad (3.2.4)$$

$$= ((((\mathfrak{q} \to (\mathfrak{p} \to \mathfrak{o})^*)^* \to \mathfrak{o})^* \to \mathfrak{o})^* \to ((\mathfrak{q} \to \mathfrak{p})^* \to \mathfrak{o})^*) \to^* (((\mathfrak{q} \to (\mathfrak{p} \to \mathfrak{o})^*)^* \to \mathfrak{o})^* \to (\mathfrak{q} \to \mathfrak{p})^*)^*)^*$$

$$= ((((\mathfrak{q} \to (\mathfrak{p} \to \mathfrak{o})^*)^* \to \mathfrak{o})^* \to \mathfrak{o})^* \to ((\mathfrak{q} \to (\mathfrak{p} \to \mathfrak{o})^*)^* \to \mathfrak{o})^*) \to (\mathfrak{q} \to \mathfrak{p})^*)^*)^* \to ((\mathfrak{q} \to \mathfrak{p})^* \to \mathfrak{o})^*)^*$$

$\leq (((ɋ \to 𝔭)^* \to ɒ)^* \to ((ɋ \to 𝔭)^* \to ɒ)^*)^* = 0$

Hence, $\left(\left(\left((ɋ \otimes (𝔭 \otimes ɒ)\right) \otimes ɒ\right) \otimes ɒ\right) \otimes ((ɋ \otimes y) \otimes ɒ)),\quad ((((ɋ \to (𝔭 \to ɒ)^*)^* \to ɒ)^* \to ɒ)^* \to ((ɋ \to 𝔭)^* \to ɒ)^*)^* \in I, (((ɋ \otimes (𝔭 \otimes ɒ)) \otimes ɒ) \otimes ɒ),\ ((((ɋ \to (𝔭 \to ɒ)^*)^* \to ɒ)^* \to ɒ)^* \in I.$

From the given cpndition, we have $((𝔭 \otimes ɒ) \otimes ɒ, ((𝔭 \to ɒ)^* \to ɒ)^* \in I.$

From the proposition 3.2.4, we have I is an associative WI-ideal.

Equation (3.2.3) comes from $ɋ \otimes 𝔭 \leq (\,𝔭 \otimes ɒ) \otimes (ɋ \otimes ɒ), (ɋ \to 𝔭) \leq (𝔭 \to ɒ) \to (ɋ \to ɒ)$

So $((ɋ \otimes ɒ) \otimes (𝔭 \otimes ɒ)) \leq ɋ \otimes 𝔭, ((ɋ \to ɒ)^* \to (𝔭 \to ɒ)^*)^* \leq (ɋ \to 𝔭)^*$ that,

$\left((ɋ \otimes ɒ) \otimes (𝔭 \otimes ɒ)\right) \otimes (ɋ \otimes 𝔭) = 0, (((ɋ \to ɒ)^* \to (𝔭 \to ɒ)^*)^* \to (ɋ \to 𝔭)^*) = 0$ and

the inequality in (3.2.3) from $(ɒ \otimes 𝔭) \leq (ɋ \otimes ɒ) \to (z \otimes 𝔭), ɒ \to 𝔭 \leq (ɋ \to ɒ) \to (ɋ \to 𝔭)$ then

$(ɋ \otimes 𝔭) \otimes (ɋ \otimes ɒ) \leq ɒ \otimes 𝔭.$

4. CONCLUSION

In this paper, we have studied positive implicative WI-ideal and associative WI-ideal of RLW-algebra and investigated some of their properties. Also, we have analyzed the relationship of positive implicative WI-ideal with implicative WI-ideal and WI-ideal, and hence an associative WI-ideal with WI-ideal. Moreover, we provide the condition equivalent for both positive implicative WI-ideal and associative WI-ideal.

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