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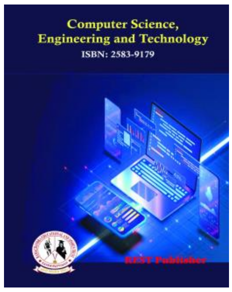

Challenges in Numerical Solutions of Higher-Dimensional Differential Equations

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Abstract. *Differential equations constitute a fundamental tool in modeling various natural phenomena across scientific disciplines such as physics, engineering, and finance. We provide an overview of fractional differential equations, focusing on the computational requirements associated with their numerical solutions from a computer science perspective. We analyze the computational intricacies concerning First-Order Linear ODE, First-Order Nonlinear ODE, Second-Order Linear ODE, Second-Order Nonlinear ODE, Heat Equation (PDE), and Wave Equation (PDE). This comparative assessment delves into the computational demands of solving these equations using differential equation methodologies. While analytical solutions provide deep insights, obtaining numerical solutions, particularly in higher dimensions, remains a persistent challenge. Finite difference methods commonly employed for numerical solutions, In higher-dimensional problems, traditional numerical methods face challenges stemming from an exponential surge in grid points and the consequent demand for substantially decreased time step sizes. This paper explores the challenges posed by higher-dimensional differential equations in numerical solutions. It highlights the infeasibility of finite difference methods in such scenarios and emphasizes the need for innovative numerical techniques capable of efficiently handling the complexities of higher-dimensional differential equations. Overcoming these challenges is crucial for advancing our understanding and modeling capabilities in complex real-world systems governed by differential equations. Continued research efforts strive to develop novel numerical methodologies capable of addressing these challenges, aiming to broaden the scope of solvable higher-dimensional differential equations and expand their application across diverse scientific domains.*
Keywords: *Fractional equations, ordinary differential equation, GRA method*

1. INTRODUCTION

Differential equations hold immense significance in mathematics and serve as crucial descriptors for physical phenomena. Attaining solutions for these equations stands as a pivotal task. While certain types of differential equations offer clear, direct analytical solutions, the advancement of computational methods has notably eased the process of solving equations. Yet, comprehending these solutions and their application in describing real-world processes can pose challenges in some scenarios. Even with the significant advancements in computational methodologies during the 21st century, there remains a persistent demand for analytical studies. This is primarily because analytical solutions generally offer a transparent understanding of the obtained solutions. For instance, leveraging expansions in series involving orthogonal polynomials has proven immensely valuable in solving numerous physical problems, as evidenced in various research references. [1], [15], and [16] are indicative of the utility and relevance of these approaches in addressing a wide array of physical problems. The variation iteration method [1-24] stands as a novel numerical approach aimed at solving both linear and nonlinear partial differential equations devoid of linearization or reliance on small perturbations. Drawing inspiration from the general Lagrange multiplier method [26], the variation iteration method primarily emerged to tackle nonlinear quandaries in quantum mechanics. Within this method, a correction function is formulated utilizing a general Lagrange multiplier, discerned through variation theory. An analytical solution is then derived from a trial function incorporating potential unknown constants, subsequently determined by progressively applying boundary conditions. Demonstrated effectiveness lies in the method's capability to effectively, readily, and accurately solve a broad spectrum of nonlinear problems, with approximations converging swiftly toward precise solutions. In contrast, the differential transform method, initially introduced by Zhou [31], adopts a different approach. This method devises an analytical solution structured in polynomial form, diverging from conventional high-order Taylor series methods that necessitate symbolic computation of requisite function derivatives. The differential transform process represents an iterative technique for attaining analytical differential equations.

The concept of fractional derivatives emerges naturally when considering differential operations like $\partial u/\partial x$ and $\partial 2u/\partial x2$; it suggests the plausibility of $\partial 1.5u/\partial x1.5$. Fractional equations offer a means to describe certain physical phenomena more precisely than classical integer order differential equations [1]. These fractional differential equations serve as a potent tool for characterizing memory and hereditary traits exhibited by diverse substances. Specifically, fractional diffusion equations hold substantial significance in various research domains such as semiconductor studies, hydrogeology, bioinformatics, finance [2], and numerous other scientific areas [3–6]. Their application proves instrumental in elucidating complex dynamics within these systems, reflecting their broad utility in modeling intricate real-world phenomena. Rajeev and Kushwaha [7] introduced a mathematical model delineating the time fractional anomalous diffusion process inherent in a generalized Stefan problem, representing a specific instance of a shoreline problem. Meanwhile, space fractional advection-diffusion equations emerge in scenarios where velocity variations exhibit heavy-tailed distributions, portraying particle movement while considering the comprehensive variations in the flow field across the entire system [8]. FDEs generally fall into two primary categories: time fractional differential equations and space fractional differential equations. Research into fractional ordinary equations and fractional order control systems has also been explored [9, 10]. Notably, the stability analysis of fractional order control systems has garnered significant attention within the academic community [11, 12]. For instance, Maione [13] delved into investigating the Laguerre continued fraction expansion of the Tustin fractional discrete-time operator. These studies contribute to a deeper understanding of the behavior and properties of systems governed by fractional calculus, showcasing diverse applications across various scientific disciplines. Absolutely, differential equations serve as foundational tools across multiple fields like physics, engineering, and finance, among others. However, obtaining numerical solutions for these equations has posed a persistent challenge. Finite difference methods, while effective in many cases, encounter limitations in higher dimensions. The significant increase in the number of grid points required in higher dimensions, coupled with the need for reduced time step sizes, often renders finite difference methods impractical or computationally demanding. This limitation prompts the exploration and development of alternative numerical techniques capable of handling the challenges posed by higher-dimensional differential equations without encountering grid and step size constraints.

2. MATERIALS AND METHOD

Alternatives (Equations):

1. *First-Order Linear ODE:*

A first-order linear ordinary differential equation (ODE) is an equation involving a derivative of a function (y) with respect to an independent variable (x) in the form:

$dy / dx\ P(x)\cdot y=Q(x)$

Here, (P(x)) and (Q(x)) are functions of (x) that determine the behavior of the differential equation.

The equation is linear because y and its derivative appear in a linear fashion (i.e., to the first power, not squared or cubed, etc.).

The general strategy to solve a first-order linear ODE involves identifying an integrating factor, often denoted as $\mu(x)$, which is a function that helps in simplifying the equation. This integrating factor is calculated as:

$\mu(x)=e^{\int P(x)dx}$

Once the integrating factor is determined, the equation is multiplied throughout by this factor to transform it into a form that allows easier integration. After simplification and integration, the general solution for $y(x)$ can be obtained. These types of differential equations are fundamental in various scientific fields, including physics, engineering, economics, and biology, among others, as they model various real-world phenomena where the rate of change of a quantity is proportional to the quantity itself, modified by other functions of the independent variable.

2. *First-Order Nonlinear ODE:*

A first-order nonlinear ordinary differential equation (ODE) involves a function $y(x)$ and its derivative dy / dx where the highest derivative present is of the first order, but the equation is nonlinear due to nonlinear relationships between the function, its derivative, and/or the independent variable x. It is expressed in the form:

$dy / dx = f(x,y)$

Here:

- dx/dy represents the first derivative of y with respect to the independent variable x.
- $f(x,y)$ is a function that can involve products, powers, trigonometric functions, exponentials, or other nonlinear terms of x and y.

Unlike linear ODEs, where the equation is a linear combination of y, dx/dy, and functions of x, nonlinear ODEs can have more complex behaviors and solutions due to the nonlinear relationships present in the equation.

Solving nonlinear ODEs generally involves various techniques depending on the specific form of the equation, such as separation of variables, substitution methods, integrating factors, or even numerical methods when analytical solutions are difficult or impossible to find. Nonlinear ODEs are extensively used in modeling complex systems in physics, biology, chemistry, engineering, economics, and many other fields where the relationships between variables are not purely proportional or additive, leading to more intricate and diverse behaviors.

3. *Second-Order Linear ODE:*

A second-order linear ordinary differential equation (ODE) involves a function $y(x)$ and its derivatives up to the second order, usually expressed in the form:

$d^2y/ dx^2+P(x)\cdot dy / dx +Q(x)\cdot y=R(x)$

Here:

- d^2y/ dx^2 represent the second derivative of y with respect to the independent variable x.
- dy /dx denotes the first derivative of y with respect to x.
- $P(x)$, $Q(x)$, and $R(x)$ are functions of x that define the behavior of the equation.

The equation is termed "linear" because y, its first derivative dy / dx, and its second derivative d^2y/ dx^2 appear in the equation to the first power, without being multiplied together or raised to other powers. However, $P(x)$, $Q(x)$, and $R(x)$ can be arbitrary functions of x.

The general solution to a second-order linear ODE involves finding particular solutions that satisfy the equation. This can be done using various methods, such as the method of undetermined coefficients, variation of parameters, or using the characteristic equation when the equation is homogeneous. These types of equations are crucial in physics and engineering, describing various natural phenomena, such as harmonic oscillations (e.g., in mechanical systems like springs), electrical circuits, vibrating systems, and more complex physical systems. They also appear in mathematical models of diverse fields like biology, economics, and chemistry.

4. *Second-Order Nonlinear ODE:*

A second-order nonlinear ordinary differential equation (ODE) involves a function $y(x)$ and its derivatives up to the second order, where the equation expresses nonlinear relationships between the function, its derivatives, and/or the independent variable x. It is typically expressed in the form:

$d^2y/ dx^2 = f(x,y, dy /dx)$

Here:

- d^2y/ dx^2 represents the second derivative of y with respect to the independent variable x.
- $f(x,y, dy/dx)$ is a function that includes nonlinear terms involving x, y, and/or dy/dx.

Unlike second-order linear ODEs, where the equation involves linear combinations of y, its derivatives, and functions of x, second-order nonlinear ODEs exhibit nonlinear relationships between these components, leading to more complex behavior and solutions.

Solving second-order nonlinear ODEs can be challenging and often involves a wide range of methods, including substitution, transforming equations into standard forms, special functions, numerical techniques, and perturbation methods, among others. Sometimes, closed-form solutions are difficult or impossible to find, especially for more complex nonlinear ODEs. Second-order nonlinear ODEs find applications in various fields, including physics (e.g., in modeling nonlinear oscillations, chaotic systems), biology (e.g., population dynamics), mechanics (e.g., in nonlinear systems of particles), and many other areas where the relationships between variables are nonlinear, giving rise to intricate and often rich behavior.

5. *Heat Equation (PDE):*

The heat equation is a partial differential equation (PDE) that describes how the distribution of heat evolves over time in a given region. It's a fundamental equation in the study of heat transfer and diffusion phenomena. The one-dimensional form of the heat equation is:

$$\partial u/\partial t = \alpha\, \partial^2 u/\partial x^2$$

Here:

- $u(x,t)$ represents the temperature distribution in space x and time t.
- $\partial u/\partial t$ denotes the rate of change of temperature with respect to time.
- $\partial^2 u/\partial x^2$ represents the second derivative of temperature with respect to space, describing how the temperature changes with position.
- α is the thermal diffusivity, a constant that characterizes the material's ability to conduct heat.

The heat equation describes how temperature changes at each point in the material over time, based on the material's ability to conduct heat. It illustrates how heat diffuses through a medium, causing the temperature to smooth out and approach an equilibrium state over time.

Solving the heat equation involves finding the temperature distribution $u(x,t)$ as a function of space and time, given initial conditions (the temperature distribution at $t=0$) and, potentially, boundary conditions that specify constraints at the boundaries of the region. The heat equation has broad applications in various fields, including physics, engineering, biology, and finance. It's used to model heat flow in materials, diffusion processes, the behavior of physical systems, and even in financial mathematics to describe the diffusion of stock prices over time.

6. *Wave Equation (PDE):*

The wave equation is a partial differential equation (PDE) that describes how waves propagate through a medium, whether it's a mechanical wave in a physical medium like a string, sound waves in air, or electromagnetic waves like light. The one-dimensional form of the wave equation is:

$$\partial^2 u/\partial t^2 = c^2 .\ \partial^2 u/\partial x^2$$

Here:

- $u(x,t)$ represents the displacement or amplitude of the wave as a function of position x and time t.

- $\partial^2 u/\partial t^2$ denotes the second derivative of u with respect to time, representing the acceleration or change in the wave's behavior over time.
- $\partial^2 u/\partial x^2$represents the second derivative of u with respect to space, describing how the wave behaves in space.
- c is the wave speed, which characterizes how fast the wave travels through the medium.

The wave equation describes how disturbances or variations in the medium propagate and evolve over time. It governs how waves move, interact, and behave in the given medium, showing how disturbances spread out and propagate.

Solving the wave equation involves finding the function $u(x,t)$ that satisfies the equation, given initial conditions (the initial displacement and velocity of the medium) and potentially boundary conditions that describe constraints at the boundaries of the medium. This equation is foundational in physics and engineering, as it's used to model various wave phenomena in different fields, including acoustics, optics, seismology, electromagnetism, and more. Understanding wave behavior through the wave equation is crucial in designing and analyzing systems that involve waves and wave-like phenomena.

Evaluation Parameters:

Analytical Solvability: Analytical solvability serves as a fundamental criterion in evaluating differential equations, dictating the ease or difficulty in finding exact, closed-form solutions without resorting to numerical approximation. A score of 1 denotes equations posing considerable challenges or those requiring complex techniques or transformations, often defying traditional analytical methods. These equations might lack known solution techniques, demanding advanced mathematical tools or remaining unsolved in many cases. Conversely, a score of 5 signifies equations that readily yield to analytical approaches, allowing straightforward application of established mathematical methods to derive precise solutions. Such equations tend to possess known solution forms or readily reducible structures, simplifying the process for mathematicians or scientists to derive explicit mathematical expressions representing their behavior. Evaluating analytical solvability aids in understanding the accessibility and tractability of differential equations, impacting their practical usability in various scientific, engineering, and mathematical contexts.

Existence and Uniqueness of Solutions: The criterion of existence and uniqueness of solutions embodies the fundamental nature of differential equations, delineating whether solutions are well-defined and singular. A score of 1 designates equations that either lack solutions entirely within the prescribed context or have multiple, conflicting solutions, leading to ambiguity or inconsistency in their interpretations. These equations might exhibit behaviors that defy predictability or fail to converge to meaningful solutions under specified conditions. Conversely, a score of 5 characterizes equations where solutions unequivocally exist and are uniquely determined for given initial or boundary conditions. Such equations reliably model systems or phenomena with consistent and predictable outcomes, showcasing the stability and reliability of their solutions. Assessing the existence and uniqueness of solutions aids in discerning the reliability and robustness of differential equations, informing practitioners about the predictability and coherence of the modeled systems or processes.

Physical Interpretation: The criterion of physical interpretation illuminates the capacity of differential equations to translate mathematical models into tangible, understandable real-world phenomena. A score of 1 characterizes equations that present challenges in elucidating their connection to physical reality, often featuring abstract or convoluted representations that hinder direct associations with observable processes. Such equations may lack immediate relevance or struggle to map onto concrete physical systems, necessitating intricate interpretations or additional context for practical understanding. Conversely, a score of 5 indicates equations that seamlessly align with intuitive and lucid physical interpretations, mirroring recognizable behaviors of natural or engineered systems with clarity and direct correspondence. These equations elegantly depict phenomena or processes, facilitating straightforward connections between mathematical formulations and observable phenomena, thereby enhancing their applicability and usability in scientific, engineering, or practical contexts. Evaluating the physical interpretation criterion aids in gauging the translational efficacy of differential equations, enabling researchers and practitioners to grasp the real-world implications and applications of mathematical models.

Computational Complexity: Computational complexity serves as a crucial criterion in assessing the practical feasibility of implementing differential equations in computational settings. A score of 1 designates equations demanding substantial computational resources, intricate algorithms, or extended processing times for their numerical solution. These equations might involve intricate nonlinearities, high-dimensional systems, or intricate structures that significantly challenge computational approaches, demanding advanced techniques or computational power. Conversely, a score of 5 signifies equations that exhibit low computational complexity, allowing for efficient numerical solutions without imposing excessive computational demands. These equations typically feature simple, manageable structures or exhibit properties amenable to streamlined computational methods, facilitating rapid computations and efficient modeling. Assessing computational complexity aids in discerning the pragmatic viability of differential equations for computational simulations or modeling, guiding researchers and practitioners in selecting equations suited for efficient computational implementation in various scientific, engineering, or technological applications.

Stability Analysis: Stability analysis stands as a pivotal criterion in evaluating the robustness and predictability of differential equation solutions. A score of 1 denotes equations yielding solutions prone to instability or intricate, challenging stability assessments. These equations might exhibit solutions that diverge rapidly, oscillate unpredictably, or necessitate sophisticated mathematical techniques for stability analysis, making predictions arduous or uncertain. Conversely, a score of 5 characterizes equations where solutions demonstrate stability and offer straightforward or easily interpretable stability analyses. Such equations often yield solutions that converge reliably, exhibit predictable behaviors, or allow for clear, understandable stability assessments without necessitating complex mathematical frameworks. Evaluating stability analysis aids in discerning the reliability and predictability of differential equation solutions, guiding practitioners in selecting equations suitable for modeling stable systems or phenomena across various scientific, engineering, or practical contexts.

GRA method:

The Grey Relational Analysis (GRA) method, initially formulated by Deng [17], has found wide application in numerous complex decision-making scenarios [24]. Rooted in grey system theory, GRA is particularly adept at resolving intricate relationships among multiple factors and variables [49]. Notably, various adaptations and modifications of the GRA method have been proposed in the existing literature [23, 25]. In our study, we introduce a simplified yet effective GRA approach, and methodology. While GRA utilizes grey relational grades to assess the correlation between reference and alternative series. This GRA method's calculation process offers a succinct yet efficient way to analyze and evaluate complex decision-making scenarios [19].

Firstly, the decision matrix is normalized in Step 1, ensuring a consistent scale across various criteria for fair comparisons. Step 2 involves computing the weighted normalized decision matrix, where criteria are assigned relative importance through weighted considerations. Subsequently, Step 3 identifies the positive ideal and negative ideal solutions, establishing benchmarks for evaluating alternatives. In Step 4, grey relational coefficients are computed based on the weighted matrix, indicating the relationship between each alternative and the positive ideal solution for each criterion.

The overall evaluation is derived in Step 5 through the calculation of the grey relational grade, representing the average of relational coefficients across all criteria. Step 6 involves determining the relative grey relational grade, offering a comparative measure of alternative performance across the overall evaluation. Finally, Step 7 involves ranking the alternatives in descending order based on their Ci values, ultimately selecting the alternative with the highest Ci as the preferred choice. This structured process aids decision-makers in systematically assessing and selecting alternatives based on multiple criteria, leveraging grey relational analysis to facilitate informed decision-making.

3. RESULT AND DISCUSSION

TABLE 1. Alternatives

First-Order Linear ODE	A1
First-Order Nonlinear ODE	A2
Second-Order Linear ODE	A3
Second-Order Nonlinear ODE	A4
Heat Equation (PDE)	A5
Wave Equation (PDE)	A6

In the realm of differential equations and partial differential equations, a diverse spectrum of mathematical models captures various phenomena. First-order linear ordinary differential equations (A1) offer a foundational understanding of linear relationships between a function and its derivative, while first-order nonlinear ODEs (A2) showcase the complexities arising from nonlinear interactions between variables. Moving to higher complexity, second-order linear ODEs (A3) describe systems where the second derivative of a function is linked to the function itself and its first derivative, often seen in harmonic oscillations. Conversely, second-order nonlinear ODEs (A4) delve into intricate dynamics, expressing nonlinear dependencies between the function, its derivatives, and the independent variable. Extending beyond ODEs, the heat equation (A5) embodies diffusion phenomena, illustrating how heat propagates through materials over time. On a similar wavelength, the wave equation (A6) characterizes the propagation of waves in diverse mediums, offering insights into how disturbances evolve in space and time. These alternatives represent a continuum of mathematical frameworks essential in modeling and understanding diverse real-world phenomena across various scientific domains.

TABLE 2. Evaluation Parameters

Analytical Solvability	C1
Existence & Uniqueness	C2
Physical Interpretation	C3
Computational Complexity	C4
Stability Analysis	C5

Table 2 shows crucial evaluation parameters for different mathematical models and equations. Analytical solvability (C1) delineates the models' amenability to explicit solutions through mathematical techniques. Existence and uniqueness (C2) denote the conditions ensuring a well-posed problem, guaranteeing the existence of a solution and its uniqueness under specific constraints. Physical interpretation (C3) signifies the capacity of equations to represent and explain real-world phenomena, providing insight into the underlying physical processes. Computational complexity (C4) gauges the level of difficulty involved in numerically solving these equations, vital for practical applications where exact solutions might be elusive. Stability analysis (C5) pertains to assessing the behavior of solutions under perturbations, crucial in understanding whether small disturbances amplify or dampen over time, affecting the system's behavior. Each parameter plays a pivotal role in discerning the applicability, usability, and reliability of mathematical models across various scientific disciplines.

TABLE 3. Data Set

	C1	C2	C3	C4	C5
A1	4	5	4	5	5
A2	2	4	3	4	3
A3	5	5	5	3	5
A4	2	3	2	4	2
A5	3	5	5	3	4
A6	**4**	**5**	**5**	**3**	**5**

The provided values in the matrix represent the scores assigned to six alternatives (A1 through A6) across five criteria (C1 to C5) in an evaluation context. Looking at the values, it's evident that there's variability in how the alternatives are rated across the different criteria. For instance, A3 consistently scores high across most criteria, obtaining top scores (5) in C1, C2, C3, and C5, indicating strong performance or alignment with the evaluative considerations in these areas. On the other hand, A2 and A4 demonstrate more moderate performances across most criteria, with scores ranging between 2 and 4, suggesting a mixed or moderate alignment with the specified evaluation parameters. Meanwhile, A5 exhibits a varying profile, with different scores across criteria, showcasing strengths in some areas (such as C2 and C5) and potential weaknesses in others (like C1 and C4). Lastly, A1 and A6 consistently perform well across several criteria, obtaining higher scores (4 and 5) in most cases, indicating their alignment or strength in meeting the evaluation criteria, particularly in C2, C4, and C5. Overall, these assigned values highlight the diverse performances of the alternatives across multiple criteria, enabling a comparative analysis to identify strengths, weaknesses, and areas for improvement across the evaluated entities.

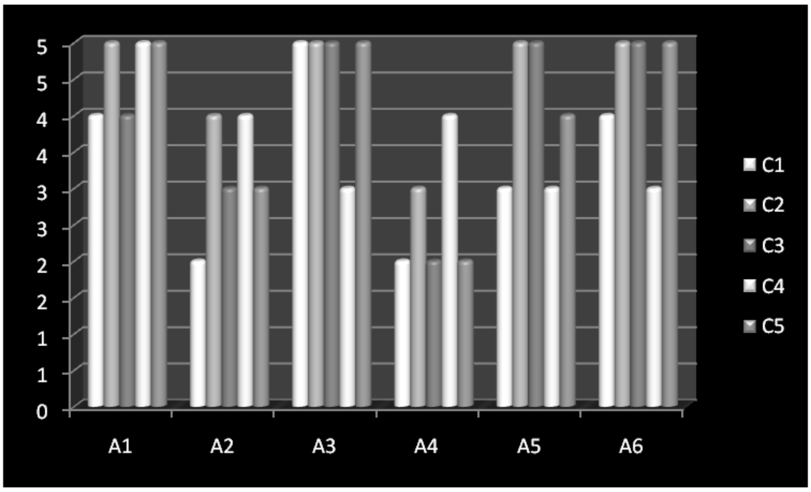

FIGURE 1. Data provides ratings

This Figure 1 data provides ratings (on a scale from 1 to 5) for each evaluation parameter (C1 to C5) corresponding to the respective mathematical models or equations (A1 to A6). These ratings can indicate the relative strengths or weaknesses of each model concerning the evaluation parameters.

TABLE 4. Normalized Data

	C1	C2	C3	C4	C5
A1	0.6667	1.0000	0.6667	0.0000	0.0000
A2	0.0000	0.5000	0.3333	0.5000	0.6667
A3	1.0000	1.0000	1.0000	1.0000	0.0000
A4	0.0000	0.0000	0.0000	0.5000	1.0000
A5	0.3333	1.0000	1.0000	1.0000	0.3333
A6	0.6667	1.0000	1.0000	1.0000	0.0000

The provided matrix displays normalized scores assigned to six alternatives (A1 through A6) across five criteria (C1 to C5). These normalized values, ranging between 0 and 1, offer insights into the relative performance of each alternative across different evaluation parameters. Alternative A3 emerges as a standout performer, obtaining a perfect score of 1 in all criteria except C5, where it receives a score of 0. This suggests that A3 excels across most dimensions but might have room for improvement in C5. Conversely, A4 consistently scores lower, achieving zeros in C1, C2, and C3, indicating potential weaknesses in these areas. This implies that A4 may not align well with the corresponding evaluation criteria. A5 showcases a diverse performance profile, scoring moderately in C1, C3, and C5 but achieving full scores in C2 and C4. This suggests specific strengths and areas for improvement for A5. A1, A2, and A6 generally perform well, with varying scores across criteria. A1 and A6 receive top scores in C2, C4, and C5, while A2 demonstrates strength in C2 and C4. The lower scores in C1 and C3 for A2 hint at potential areas for enhancement. In summary, the normalized scores enable a nuanced understanding of each alternative's performance, facilitating a comprehensive comparison and aiding decision-makers in identifying strengths, weaknesses, and potential optimization opportunities across the evaluated criteria.

TABLE 5. Deviation sequence

	C1	C2	C3	C4	C5
A1	0.3333	0.0000	0.3333	1.0000	1.0000
A2	1.0000	0.5000	0.6667	0.5000	0.3333
A3	0.0000	0.0000	0.0000	0.0000	1.0000
A4	1.0000	1.0000	1.0000	0.5000	0.0000
A5	0.6667	0.0000	0.0000	0.0000	0.6667
A6	0.3333	0.0000	0.0000	0.0000	1.0000

The provided table illustrates deviation sequences for six alternatives (A1 through A6) across five criteria (C1 to C5). These deviation values signify the extent of divergence or difference of each alternative from an ideal or benchmark performance in each criterion. Alternative A3 displays a deviation sequence indicating consistently no deviation (0.0000) across all criteria except for C5, where it deviates fully (1.0000). This suggests that A3 closely adheres to the benchmark in most criteria but significantly deviates in C5. Alternatives A2 and A4 exhibit high deviation values (mostly 1.0000) across multiple criteria, indicating significant divergence from the benchmark performance. A2 particularly diverges in C1 and C5, while A4 showcases notable deviations in C1, C2, and C3. Alternatives A1, A5, and A6 show more varied deviation sequences. A1 demonstrates notable deviations in C4 and C5, A5 diverges primarily in C1 and C5, while A6 showcases substantial deviation in C5. Overall, these deviation sequences offer insights into the extent of divergence of each alternative from an ideal or benchmark performance in different evaluation criteria, aiding in identifying areas of excellence or improvement across the evaluated dimensions.

TABLE 6. Grey relation coefficient

	C1	C2	C3	C4	C5
A1	0.6000	1.0000	0.6000	0.3333	0.3333
A2	0.3333	0.5000	0.4286	0.5000	0.6000
A3	1.0000	1.0000	1.0000	1.0000	0.3333
A4	0.3333	0.3333	0.3333	0.5000	1.0000
A5	0.4286	1.0000	1.0000	1.0000	0.4286
A6	0.6000	1.0000	1.0000	1.0000	0.3333

The provided table presents the Grey Relation Coefficients for six alternatives (A1 through A6) across five criteria (C1 to C5). These coefficients reflect the degree of correlation or similarity between each alternative and an ideal reference within each criterion. Alternative A3 consistently obtains high coefficients of 1.0000 across all criteria except for C5, where it scores 0.3333. This indicates that A3 closely resembles the ideal reference in most criteria but displays some divergence in C5. Alternatives A1, A2, A5, and A6 exhibit varied coefficients across the criteria. A1 showcases strong correlations in C2 and C3, while A2 demonstrates notable correlations in C2 and C5. A5 displays strong correlations in C2, C3, and C4, and A6 shows high correlations in C2, C3, and C4 as well. Alternative A4 showcases lower coefficients across most criteria, particularly in C1, C2, and C3, suggesting comparatively weaker correlations or similarities with the ideal reference across these dimensions. Overall, these coefficients provide insights into the degree of resemblance or proximity of each alternative to an ideal reference within each criterion, aiding in the comparative analysis and identification of alternatives with higher correlation to the ideal benchmark within the evaluated dimensions.

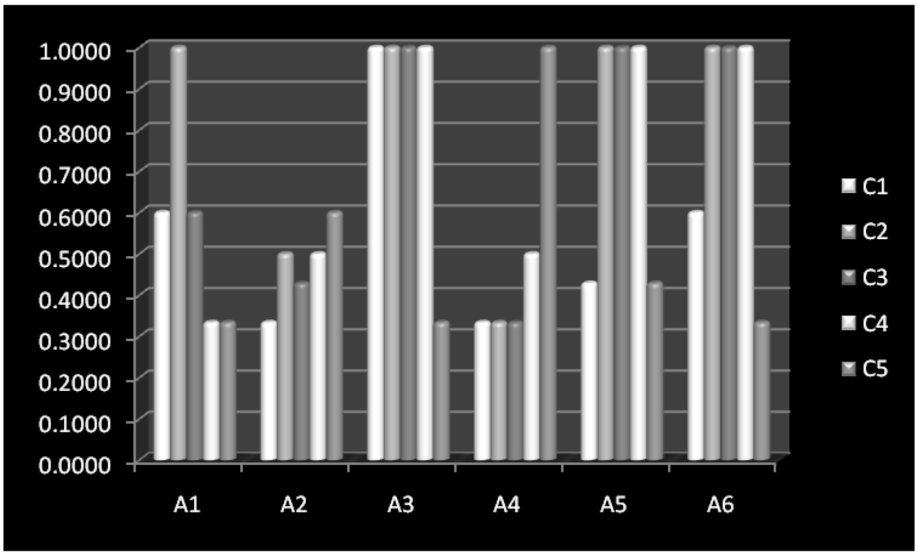

FIGURE 2. Grey relation coefficient

The Figure 2 grey relation analysis provides a quantitative perspective on the relative merits of each mathematical model, aiding in the assessment and selection of models based on their performance across multiple dimensions.

TABLE 7. GRG

A1	0.5733
A2	0.4724
A3	0.8667
A4	0.5000
A5	0.7714
A6	0.7867

The GRG (Grey Relational Grade) values presented in Table 7 denote the overall performance or grade of six alternatives (A1 through A6) across multiple criteria, consolidating the grey relation coefficients into a single aggregated score for each alternative. A3 emerges as the top performer with a GRG value of 0.8667, signifying its high overall grade compared to other alternatives. This suggests that A3 exhibits the closest correlation or similarity to the ideal reference across the evaluated criteria, consolidating strong performance across multiple dimensions. A5 and A6 follow closely behind with GRG values of 0.7714 and 0.7867, respectively, showcasing robust performances across the criteria but slightly lower than A3. Alternatives A1 and A4 demonstrate intermediate GRG values of 0.5733 and 0.5000, indicating moderately favorable performances but falling behind A3, A5, and A6 in terms of overall grade. A2 obtains the lowest GRG value of 0.4724 among the alternatives, suggesting a comparatively weaker overall performance or correlation with the ideal benchmark across the evaluated criteria. Overall, these GRG values offer a consolidated assessment of the overall performance of each alternative, aiding in the comparative analysis and ranking of alternatives based on their aggregated performance across multiple dimensions.

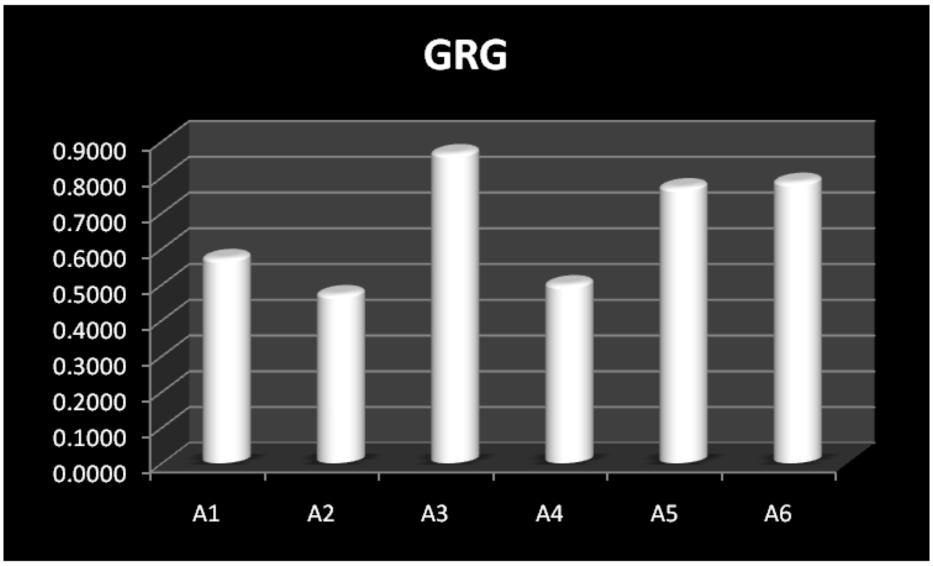

FIGURE 3. GRG values

The Figure 3 GRG values offer a concise representation of each model's overall performance, aiding in the comparative assessment and selection of models based on their aggregated performance across multiple evaluation dimensions.

TABLE 8. Rank

First-Order Linear ODE	4
First-Order Nonlinear ODE	6
Second-Order Linear ODE	1
Second-Order Nonlinear ODE	5
Heat Equation (PDE)	3
Wave Equation (PDE)	2

In the provided ranking of different types of differential equations (DEs), the "Second-Order Linear ODE" claims the top spot, implying its superior performance or alignment with the specified evaluation criteria compared to other listed DE types. This suggests that within the evaluated context or criteria, the Second-Order Linear ODE excelled the most, showcasing its strength or compatibility with the outlined benchmarks. Following closely behind is the "Wave Equation (PDE)" and the "Heat Equation (PDE)," securing the second and third positions, respectively. These equations demonstrated strong performances but slightly below the top-ranked Second-Order Linear ODE. Meanwhile, the "First-Order Linear ODE" occupies the fourth position, indicating a respectable but comparatively less impressive performance within the given criteria. The "Second-Order Nonlinear ODE" and the "First-Order Nonlinear ODE" secure the fifth and sixth positions, respectively, suggesting their relatively weaker performances compared to other DE types in the assessed context or criteria. This ranking aids in discerning the differential equation types based on their relative standings within the specified evaluation criteria, providing insights into their comparative strengths or weaknesses within the evaluated context.

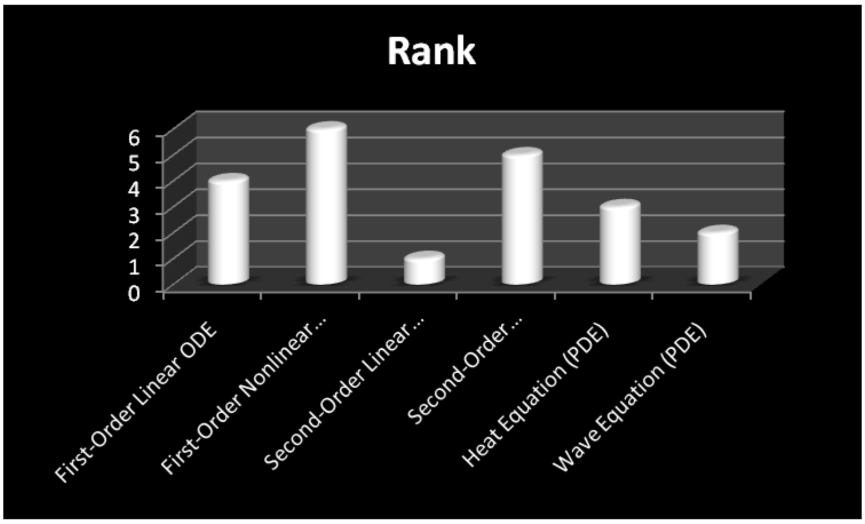

FIGURE 4. Rank

This ranking places the "Second-Order Linear ODE" at the top position, suggesting it performed the best among the listed types of differential equations within the evaluated criteria or context. Conversely, the "First-Order Nonlinear ODE" received the lowest rank, indicating relatively poorer performance compared to other types based on the evaluation criteria used.

4. CONCLUSION

In conclusion, the realm of differential equations stands as a cornerstone in diverse scientific disciplines, playing a pivotal role in modeling and understanding natural phenomena in physics, engineering, finance, and beyond. While their analytical solutions offer profound insights, the pursuit of numerical solutions has presented enduring challenges, particularly in higher dimensions. Finite difference methods, commonly employed for numerical solutions, face limitations due to the exponential increase in grid points and the necessity for smaller time step sizes in higher-dimensional problems. This challenge necessitates ongoing exploration and innovation in numerical methods to surmount the limitations of traditional approaches. Overcoming these hurdles in higher-dimensional spaces requires the development of novel techniques capable of efficiently and accurately handling complex differential equations without succumbing to computational inefficiencies. The quest for robust numerical methods remains crucial to advance our capacity to model, simulate, and comprehend complex real-world systems governed by differential equations. Thus, ongoing research endeavors aim to innovate numerical approaches that can effectively tackle the computational demands posed by higher-dimensional differential equations, furthering our understanding and application of these fundamental mathematical models across various scientific domains.

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